Appendix

A Elements of Functional Analysis

We collect in this section a number of notions and results from the theory of linear operators in Banach spaces. We refer for instance to the book [76] for further details and proofs. In addition, in Section A.6 we give the definitions and some basic results on Sobolev spaces, for which we refer to the books [1, 2].

A.1 Bounded and Closed Operators

Consider the Banach spaces $\mathscr{X}$ and $\mathscr{Z}$ equipped with the norms $\|\cdot\|_{\mathscr{X}}$ and $\|\cdot\|_{\mathscr{Z}}$, respectively, and a linear map (or linear operator) $\mathbf{L}:\mathscr{Z}\to\mathscr{X}$. We denote by $\operatorname{im}\mathbf{L}$ the *range* of $\mathbf{L}$,

$$\operatorname{im}\mathbf{L} = \{\mathbf{L}u\in\mathscr{X}\ ;\ u\in\mathscr{Z}\}\subset\mathscr{X},$$

by $\ker\mathbf{L}$ its *kernel*,

$$\ker\mathbf{L} = \{u\in\mathscr{Z}\ ;\ \mathbf{L}u=0\}\subset\mathscr{Z},$$

and by $\mathrm{G}(\mathbf{L})$ its *graph*,

$$\mathrm{G}(\mathbf{L}) = \{(u,\mathbf{L}u)\ ;\ u\in\mathscr{Z}\}\subset\mathscr{Z}\times\mathscr{X}.$$

Definition A.1 (Bounded operator) *A linear operator* $\mathbf{L}:\mathscr{Z}\to\mathscr{X}$ *is called* a bounded linear operator*, or simply* a bounded operator*, if* $\mathbf{L}$ *is continuous. The set of bounded linear operators is denoted by* $\mathscr{L}(\mathscr{Z},\mathscr{X})$*, and by* $\mathscr{L}(\mathscr{X})$ *if* $\mathscr{Z}=\mathscr{X}$.

Properties A.2 [76, Chapter III, §2.2, §3.1]

(i) For a linear operator $\mathbf{L}:\mathscr{Z}\to\mathscr{X}$ the following properties are equivalent:

a. $\mathbf{L}$ is continuous, i.e., $\mathbf{L}$ is a bounded linear operator;
b. $\mathbf{L}$ is continuous in 0;
c. $\sup\{\|\mathbf{L}u\|_{\mathscr{X}}\ ;\ u\in\mathscr{Z},\ \|u\|_{\mathscr{Z}}=1\}<\infty$.

M. Haragus, G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*, Universitext,
DOI 10.1007/978-0-85729-112-7, © EDP Sciences 2011

(ii) For a bounded linear operator $\mathbf{L}$, the real number

$$\|\mathbf{L}\|_{\mathscr{L}(\mathscr{X},\mathscr{X})} \stackrel{\text{def}}{=} \sup_{\|u\|_{\mathscr{X}}=1} \|\mathbf{L}u\|_{\mathscr{X}} = \sup_{0<\|u\|_{\mathscr{X}}\leq 1} \frac{\|\mathbf{L}u\|_{\mathscr{X}}}{\|u\|_{\mathscr{X}}} = \sup_{\|u\|_{\mathscr{X}}\neq 0} \frac{\|\mathbf{L}u\|_{\mathscr{X}}}{\|u\|_{\mathscr{X}}}$$

is called *norm* of $\mathbf{L}$.

(iii) The set of bounded linear operators $\mathscr{L}(\mathscr{X},\mathscr{X})$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathscr{L}(\mathscr{X},\mathscr{X})}$.

Definition A.3 (Closed operator) *A linear operator* $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ *defined on a linear subspace* $D(\mathbf{L}) \subset \mathscr{X}$ *is called* a closed linear operator*, or simply* a closed operator*, if its graph* $G(\mathbf{L})$ *is a closed set in* $\mathscr{X} \times \mathscr{X}$*. The set of closed linear operators is denoted by* $\mathscr{C}(\mathscr{X})$*.*

Properties A.4 [76, Chapter III, §5.2, Theorem 5.20]

(i) A linear operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ is closed if and only if for any sequence $(u_n)_{n\in\mathbb{N}} \subset D(\mathbf{L})$ such that $u_n \to u$ in $\mathscr{X}$ and $\mathbf{L}u_n \to v$ in $\mathscr{X}$, we have that $u \in D(\mathbf{L})$ and $\mathbf{L}u = v$.

(ii) The sum of a closed operator with a bounded operator is a closed operator. However, the sum of two closed operators is not always a closed operator.

(iii) A closed operator with domain $D(\mathbf{L}) = \mathscr{X}$ is bounded (closed graph theorem).

(iv) For a closed operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ the domain $D(\mathbf{L})$ equipped with the norm

$$\|u\|_{\mathbf{L}} = \left(\|u\|_{\mathscr{X}}^2 + \|\mathbf{L}u\|_{\mathscr{X}}^2\right)^{1/2}$$

is a Banach space, and the injection $i : D(\mathbf{L}) \to \mathscr{X}$ is bounded. This norm is also called *the graph norm*.

(v) A closed operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ belongs to $\mathscr{L}(D(\mathbf{L}),\mathscr{X})$ when $D(\mathbf{L})$ is equipped with the graph norm $\|\cdot\|_{\mathbf{L}}$.

A.2 Resolvent and Spectrum

Definition A.5 *Consider a linear operator* $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$*.*

(i) We call resolvent set *of* $\mathbf{L}$ *the set of complex numbers*

$$\rho(\mathbf{L}) = \{\lambda \in \mathbb{C}\,;\ (\lambda\mathbb{I} - \mathbf{L}) \textit{ invertible and } (\lambda\mathbb{I} - \mathbf{L})^{-1} \in \mathscr{L}(\mathscr{X})\}.$$

The operator $(\lambda\mathbb{I} - \mathbf{L})^{-1}$*, for* $\lambda \in \rho(\mathbf{L})$*, is called* the resolvent *of* $\mathbf{L}$.

(ii) We call the spectrum *of* $\mathbf{L}$ *the complement of the resolvent set,*

$$\sigma(\mathbf{L}) = \mathbb{C} \setminus \rho(\mathbf{L}).$$

(iii) A complex number $\lambda \in \mathbb{C}$ *is called* an eigenvalue *of* $\mathbf{L}$ *if* $\ker(\lambda\mathbb{I} - \mathbf{L}) \neq \{0\}$. *The kernel* $\ker(\lambda\mathbb{I} - \mathbf{L})$ *is called* the eigenspace *associated with the eigenvalue*

λ, *and any element* $u \in \ker(\lambda\mathbb{I} - \mathbf{L})\setminus\{0\}$ *is called* an eigenvector *associated with the eigenvalue* λ.

(iv) For an eigenvalue $\lambda \in \sigma(\mathbf{L})$, *the dimension of* $\ker(\lambda\mathbb{I} - \mathbf{L})$ *is called* the geometric multiplicity *of* λ. *An eigenvalue with geometric multiplicity one is called* geometrically simple.

(v) For an isolated eigenvalue $\lambda \in \sigma(\mathbf{L})$, *the dimension of the largest subspace* $\mathscr{X}_\lambda \subset D(\mathbf{L})$ *which is invariant under the action of* $\mathbf{L}$ *and such that* $\sigma(\mathbf{L}|_{\mathscr{X}_\lambda}) = \{\lambda\}$ *is called* the algebraic multiplicity *of* λ. *An eigenvalue with algebraic multiplicity one is called* algebraically simple *or* simple.

(vi) An eigenvalue is called semisimple *if its algebraic and geometric multiplicities are the same.*

Properties A.6 [76, Chapter III, §6.1, §6.3]

(i) The spectrum of a closed operator $\mathbf{L} \in \mathscr{C}(\mathscr{X})$ is a closed set.

(ii) The spectrum of a bounded operator $\mathbf{L} \in \mathscr{L}(\mathscr{X})$ is a closed, bounded, nonempty set.

(iii) For a closed operator $\mathbf{L} \in \mathscr{C}(\mathscr{X})$, if $\lambda \in \rho(\mathbf{L})$, then the resolvent $(\lambda\mathbb{I} - \mathbf{L})^{-1} : \mathscr{X} \to D(\mathbf{L})$ is a bounded operator, when $D(\mathbf{L})$ is equipped with the graph norm, i.e., $(\lambda\mathbb{I} - \mathbf{L})^{-1} \in \mathscr{L}(\mathscr{X}, D(\mathbf{L}))$.

(iv) The map $\lambda \mapsto (\lambda\mathbb{I} - \mathbf{L})^{-1}$ is holomorphic from $\rho(\mathbf{L})$ into $\mathscr{L}(\mathscr{X})$.

(v) For $\lambda \in \rho(\mathbf{L})$, the resolvent $(\lambda\mathbb{I} - \mathbf{L})^{-1} : \mathscr{X} \to D(\mathbf{L})$ commutes with $\mathbf{L}$, i.e.,

$$\mathbf{L}(\lambda\mathbb{I} - \mathbf{L})^{-1}u = (\lambda\mathbb{I} - \mathbf{L})^{-1}\mathbf{L}u \text{ for all } u \in D(\mathbf{L}).$$

(vi) For $\lambda, \mu \in \rho(\mathbf{L})$, the resolvents $(\lambda\mathbb{I} - \mathbf{L})^{-1}$ and $(\mu\mathbb{I} - \mathbf{L})^{-1}$ commute and

$$(\lambda\mathbb{I} - \mathbf{L})^{-1} - (\mu\mathbb{I} - \mathbf{L})^{-1} = (\mu - \lambda)(\lambda\mathbb{I} - \mathbf{L})^{-1}(\mu\mathbb{I} - \mathbf{L})^{-1}.$$

(vii) For an operator $\mathbf{L}$, we call *the extended spectrum* the set $\sigma_\infty(\mathbf{L}) \subset \mathbb{C} \cup \{\infty\}$ defined by

$$\sigma_\infty(\mathbf{L}) = \begin{cases} \sigma(\mathbf{L}) & \text{if } \sigma(\mathbf{L}) \text{ is bounded,} \\ \sigma(\mathbf{L}) \cup \{\infty\} & \text{if } \sigma(\mathbf{L}) \text{ is unbounded.} \end{cases}$$

Then for $\lambda, \mu \in \rho(\mathbf{L})$ the spectrum of the resolvent satisfies

$$\sigma\left((\lambda\mathbb{I} - \mathbf{L})^{-1}\right) = \left\{(\lambda - \mu)^{-1}\,;\, \mu \in \sigma_\infty(\mathbf{L})\right\}. \tag{A.1}$$

Theorem A.7 (Spectral decomposition [76, Chapter III, Theorem 6.17]) *Consider a closed operator* $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$. *Assume that* $\sigma(\mathbf{L}) = F \cup G$, *with* $F \cap G = \varnothing$ *and* $F \subset \mathbb{C}$ *a closed, bounded set, such that there exists a rectifiable, simple, closed curve* Γ *which encloses an open set containing* F *in its interior and* G *in its exterior. Then there exists a decomposition of* $\mathscr{X} = \mathscr{X}_F \oplus \mathscr{X}_G$, *with* $\mathscr{X}_F$ *and* $\mathscr{X}_G$ *invariant under the action of* $\mathbf{L}$, *such that the spectra of the restrictions* $\mathbf{L}|_{\mathscr{X}_F}$ *and* $\mathbf{L}|_{\mathscr{X}_G}$ *coincide with* F *and* G, *respectively. Furthermore,* $\mathbf{L}|_{\mathscr{X}_F}$ *is a bounded operator,* $\mathbf{L}|_{\mathscr{X}_F} \in \mathscr{L}(\mathscr{X}_F)$, *and the unique spectral projection* $\mathbf{P}_F : \mathscr{X} \to \mathscr{X}_F$ *which* *commutes with* $\mathbf{L}$ *is given by the Dunford integral formula*

$$\mathbf{P}_F = \frac{1}{2\pi i}\int_\Gamma (\lambda\mathbb{I} - \mathbf{L})^{-1} d\lambda.$$

Remark A.8 *(i) The result in the above theorem still holds when the curve Γ is replaced by a finite number of rectifiable, simple, closed curves.*

(ii) In the particular case when $F = \{\lambda\}$ is reduced to one point, λ is an isolated point of the spectrum. If the dimension of $\mathscr{X}_F$ is finite, then λ is an eigenvalue of $\mathbf{L}$*, and the dimension of $\mathscr{X}_F$ is the algebraic multiplicity of λ [76, Chapter III §6.5].*

A.3 Compact Operators and Operators with Compact Resolvent

Definition A.9 (Compact operator) *A linear operator $\mathbf{L} : \mathscr{Z} \to \mathscr{X}$ is called* a compact operator *if for any bounded sequence $(u_n)_{n\in\mathbb{N}} \subset \mathscr{Z}$, the sequence $(\mathbf{L}u_n)_{n\in\mathbb{N}} \subset \mathscr{X}$ contains a convergent subsequence.*

Properties A.10 [76, Chapter III §4.2]

(i) A compact operator is bounded.
(ii) The sum of two compact operators is a compact operator. For a bounded operator $\mathbf{L} : \mathscr{Y} \to \mathscr{Z}$ and a compact operator $\mathbf{K} : \mathscr{Z} \to \mathscr{X}$, the composed operator $\mathbf{K} \circ \mathbf{L} : \mathscr{Y} \to \mathscr{X}$ is compact. A similar property holds for $\mathbf{L} \circ \mathbf{K}$ (adapt the spaces).

Theorem A.11 (Spectrum [76, Chapter III, Theorem 6.26]) *Consider a compact operator $\mathbf{L} : \mathscr{X} \to \mathscr{X}$. Then the following properties hold:*

(i) $0 \in \sigma(\mathbf{L})$ if $\mathscr{X}$ is infinite-dimensional;
(ii) any $\lambda \in \sigma(\mathbf{L})$ with $\lambda \neq 0$ is an isolated eigenvalue with finite algebraic multiplicity;
(iii) $\sigma(\mathbf{L})$ is a countable set with at most one accumulation point in 0.

Definition A.12 (Operator with compact resolvent) *A linear operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ is called* an operator with compact resolvent *if $\rho(\mathbf{L}) \neq \varnothing$ and for some $\lambda \in \rho(\mathbf{L})$ the operator $(\lambda\mathbb{I} - \mathbf{L})^{-1} : \mathscr{X} \to \mathscr{X}$ is compact.*

Properties A.13 [76, Chapter III §6.8]

(i) If $(\lambda\mathbb{I} - \mathbf{L})^{-1} : \mathscr{X} \to \mathscr{X}$ is a compact operator for some $\lambda \in \rho(\mathbf{L})$, then it is a compact operator for any $\lambda \in \rho(\mathbf{L})$.
(ii) The spectrum of an operator with compact resolvent is a countable set consisting of isolated eigenvalues with finite algebraic multiplicities, with no accumulation point in $\mathbb{C}$.

A.4 Adjoint Operator

For a Banach space $\mathscr{X}$, denote by $\mathscr{X}^*$ the dual space, i.e., the space of all continuous linear forms on $\mathscr{X}$, and by $\langle\cdot,\cdot\rangle$ the duality product defined by

$$\langle u, u^* \rangle = \overline{u^*}(u) \text{ for all } u \in \mathscr{X},\ u^* \in \mathscr{X}^*.$$

Recall that $\mathscr{X} = \mathscr{X}^*$ when $\mathscr{X}$ is a Hilbert space.

Definition A.14 (Adjoint operators) *(i) Two linear operators $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ and $\mathbf{M} : D(\mathbf{M}) \subset \mathscr{X}^* \to \mathscr{X}^*$ are called* adjoint *to each other if*

$$\langle \mathbf{L}u, v \rangle = \langle u, \mathbf{M}v \rangle \textit{ for all } u \in D(\mathbf{L}),\ v \in D(\mathbf{M}).$$

(ii) If there exists a unique maximal operator $\mathbf{L}^*$ which is adjoint to $\mathbf{L}$, then $\mathbf{L}^*$ is called the adjoint of $\mathbf{L}$.

Properties A.15 ([76, Chapter III, §5.5, Theorem 6.22])

(i) If $D(\mathbf{L})$ is dense in $\mathscr{X}$, for an operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ there is a unique adjoint operator, but this property is not true in general. The adjoint operator $\mathbf{L}^* : D(\mathbf{L}^*) \subset \mathscr{X}^* \to \mathscr{X}^*$ is constructed in the following way. The domain $D(\mathbf{L}^*)$ consists of all $v \in \mathscr{X}^*$ such that $u \mapsto \langle \mathbf{L}u, v \rangle$ is a continuous linear form on $\mathscr{X}$. Then there exists $w \in \mathscr{X}^*$ such that

$$\langle \mathbf{L}u, v \rangle = \langle u, w \rangle \text{ for all } u \in D(\mathbf{L}),$$

and w is unique because $D(\mathbf{L})$ is dense in $\mathscr{X}$. We define $\mathbf{L}^* v = w$.

(ii) For an operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ with $D(\mathbf{L})$ dense in $\mathscr{X}$, the adjoint operator $\mathbf{L}^* : D(\mathbf{L}^*) \subset \mathscr{X}^* \to \mathscr{X}^*$ is closed. In addition, if $\mathbf{L}$ is closed and the Banach space $\mathscr{X}$ is reflexive, then $\mathbf{L}^*$ is densely defined, i.e., $D(\mathbf{L}^*)$ is dense in $\mathscr{X}^*$.

(iii) For an operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ with $D(\mathbf{L})$ dense in $\mathscr{X}$, we have

$$\ker \mathbf{L}^* = (\operatorname{im} \mathbf{L})^\perp \stackrel{\text{def}}{=} \{ v \in \mathscr{X}^* \,;\, \langle u, v \rangle = 0 \text{ for all } u \in \operatorname{im} \mathbf{L} \}.$$

(iv) For an operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ the resolvent set and the spectrum of the adjoint operator $\mathbf{L}^* : D(\mathbf{L}^*) \subset \mathscr{X}^* \to \mathscr{X}^*$ satisfy

$$\rho(\mathbf{L}^*) = \overline{\rho(\mathbf{L})}, \quad \sigma(\mathbf{L}^*) = \overline{\sigma(\mathbf{L})}.$$

Furthermore,

$$(\overline{\lambda}\mathbb{I} - \mathbf{L}^*)^{-1} = \left((\lambda \mathbb{I} - \mathbf{L})^{-1} \right)^* \text{ for all } \lambda \in \rho(\mathbf{L}).$$

Definition A.16 (Self-adjoint operator) *A linear operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ in a Hilbert space $\mathscr{X}$, with domain $D(\mathbf{L})$ dense in $\mathscr{X}$, is called self-adjoint if its adjoint $\mathbf{L}^* : D(\mathbf{L}^*) \subset \mathscr{X} \to \mathscr{X}$ satisfies $D(\mathbf{L}) = D(\mathbf{L}^*)$ and $\mathbf{L}u = \mathbf{L}^* u$ for all $u \in D(\mathbf{L})$.*

Properties A.17 **[76, Chapter V, §3.4, §3.5]**

(i) The spectrum of a self-adjoint operator is real.
(ii) The algebraic and geometric multiplicities of an isolated eigenvalue $\lambda \in \sigma(\mathbf{L})$ of a self-adjoint operator are the same, i.e., the eigenvalue is semisimple.

A.5 Fredholm Operators

Definition A.18 *A bounded operator* $\mathbf{L} \in \mathscr{L}(\mathscr{Z}, \mathscr{X})$ *is called* a Fredholm operator *if its kernel* $\ker \mathbf{L}$ *is finite-dimensional and its range* $\operatorname{im} \mathbf{L}$ *is closed and has finite codimension. The integer*

$$\operatorname{ind}(\mathbf{L}) \stackrel{def}{=} \dim(\ker \mathbf{L}) - \operatorname{codim}(\operatorname{im} \mathbf{L}),$$

is called the Fredholm index.

Remark A.19 *The above definition is easily extended to closed operators* $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$*, since* $\mathbf{L} \in \mathscr{L}(D(\mathbf{L}), \mathscr{X})$*.*

Properties A.20 **[76, Chapter IV, §5.1, Theorem 5.22, Theorem 5.28, §5.2]**

(i) The set of Fredholm operators is open in $\mathscr{L}(\mathscr{Z}, \mathscr{X})$ and the map $\mathbf{L} \mapsto \operatorname{ind}(\mathbf{L})$ is continuous.
(ii) For a closed operator $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$, if $\lambda \in \sigma(\mathbf{L})$ is an eigenvalue with finite algebraic multiplicity which is isolated in the spectrum of $\mathbf{L}$, then $\lambda \mathbb{I} - \mathbf{L}$ is a Fredholm operator with index 0.
(iii) If $\mathbf{L} : D(\mathbf{L}) \subset \mathscr{X} \to \mathscr{X}$ is a densely defined closed operator, then $\mathbf{L}$ is a Fredholm operator if and only if $\mathbf{L}^*$ is a Fredholm operator. Furthermore,

$$\operatorname{ind}(\mathbf{L}^*) = -\operatorname{ind}(\mathbf{L}).$$

A.6 Basic Sobolev Spaces

We recall in this section some basic properties of the Sobolev spaces $L^2(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}^*$ which are used in this book. We refer to [1, Chapters II, III, V, and VI] for more general statements, and for the case of the spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$.

The Space $L^2(\Omega)$

Consider a domain $\Omega \subset \mathbb{R}^n$, and the space of complex-valued square-integrable functions on Ω

$$\mathscr{L}^2(\Omega) = \left\{ f : \Omega \to \mathbb{C}\ ;\ f \text{ measurable and } \int_\Omega |f(x)|^2 dx < \infty \right\}.$$

This set is a linear space with respect to the natural operations (sum and multiplication by a complex number). For $f \in \mathscr{L}^2(\Omega)$, we set

$$\|f\| = \left(\int_\Omega |f(x)|^2 dx \right)^{1/2}. \tag{A.2}$$

Then $\|\cdot\|$ is a seminorm on $\mathscr{L}^2(\Omega)$, but not a norm, since if $\|f\| = 0$, then $f = 0$ almost everywhere, only. However, $\|\cdot\|$ can be transformed into a norm by replacing $\mathscr{L}^2(\Omega)$ by the quotient with respect to the kernel of $\|\cdot\|$, i.e., by

$$L^2(\Omega) = \mathscr{L}^2(\Omega)/\ker(\|\cdot\|).$$

Clearly, the kernel of $\|\cdot\|$ consists of functions that are equal to 0 almost everywhere, so that $L^2(\Omega)$ consists of classes of functions that are equal almost everywhere. Then $\|\cdot\|$ is a norm on $L^2(\Omega)$, or, in other words *$L^2(\Omega)$ is a normed space with norm $\|\cdot\|$ defined by (A.2)*. Furthermore, this norm corresponds to the scalar product defined by

$$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)} dx \text{ for all } f,\ g \in L^2(\Omega). \tag{A.3}$$

A key property of the space $L^2(\Omega)$ is that it is complete; more precisely we have the following result.

Properties A.21 The space $L^2(\Omega)$ equipped with the scalar product $\langle \cdot, \cdot \rangle$ defined by (A.3) is a Hilbert space.

The Spaces $H^m(\Omega)$ and $H_0^m(\Omega)$

Definition A.22 *Consider $m \in \mathbb{N}^*$.*

(i) We define the space

$$H^m(\Omega) = \left\{ u \in L^2(\Omega)\ ;\ D^\alpha u \in L^2(\Omega) \textit{ for all } \alpha \in \mathbb{N}^n,\ |\alpha| \le m \right\},$$

in which $D^\alpha u$ is the distributional partial derivative of u,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

(ii) On $H^m(\Omega)$ we define the scalar product

$$\langle f, g \rangle_m = \sum_{|\alpha| \le m} \langle D^\alpha f, D^\alpha g \rangle,$$

where $\langle\cdot,\cdot\rangle$ represents the scalar product in $L^2(\Omega)$, and the corresponding norm

$$\|u\|_m = \langle u,u\rangle_m^{1/2}.$$

(iii) We define $H_0^m(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, where $C_0^\infty(\Omega)$ is the space of functions of class C^∞ which have compact support in Ω.

Properties A.23 [1, Theorem 3.2]

(i) $H^m(\Omega)$ equipped with the scalar product $\langle\cdot,\cdot\rangle_m$ is a Hilbert space.
(ii) $H^{m+j}(\Omega)$ is a dense subspace of $H^m(\Omega)$ and the imbedding $H^{m+j}(\Omega) \hookrightarrow H^m(\Omega)$ is continuous, for any $j \in \mathbb{N}^*$.

Properties A.24 (Sobolev imbedding theorem [1, Theorem 5.4]) Assume that either $\Omega = \mathbb{R}^n$ or Ω is a bounded domain in $\mathbb{R}^n$ having a locally Lipschitz boundary, i.e., for each point x on the boundary $\partial\Omega$ of Ω there exists a neighborhood U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz continuous function.

(i) For any $j \in \mathbb{N}^*$ such that $j > n/2$, we have that $H^{m+j}(\Omega) \subset C^m(\overline{\Omega})$, and the imbedding is continuous.
(ii) If Ω is an arbitrary domain in $\mathbb{R}^n$, the result (i) holds with $H_0^{m+j}(\Omega)$ instead of $H^{m+j}(\Omega)$.

Properties A.25 (Rellich–Kondrachov theorem [2, Theorem 3.8], [1, Theorem 6.2]) Assume that Ω is a bounded domain in $\mathbb{R}^n$ having a locally Lipschitz boundary.

(i) For any $j \in \mathbb{N}^*$, the imbedding $H^{m+j}(\Omega) \subset H^m(\Omega)$ is compact.
(ii) For any $j \in \mathbb{N}^*$ such that $j > n/2$, the imbedding $H^{m+j}(\Omega) \subset C^m(\overline{\Omega})$ is compact.
(iii) If Ω is an arbitrary domain in $\mathbb{R}^n$, the results (i) and (ii) hold with $H_0^{m+j}(\Omega)$ instead of $H^{m+j}(\Omega)$.

Spaces of Periodic Functions

An important particular case is that of spaces of periodic functions on the real line. Consider the space $L^2_{\rm loc}(\mathbb{R})$ of measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f \in L^2(a,b)$ for any bounded interval $(a,b) \subset \mathbb{R}$. We define the space of square-integrable ℓ-periodic functions by

$$L^2_{\rm per}(0,\ell) = \{f \in L^2_{\rm loc}(\mathbb{R})\ ;\ f(\cdot+\ell) = f(\cdot)\},$$

and for $m \in \mathbb{N}^*$, the Sobolev spaces H^m consisting of ℓ-periodic functions by

$$H^m_{\rm per}(0,\ell) = \{f \in L^2_{\rm per}(\mathbb{R})\ ;\ f^{(k)} \in L^2_{\rm per}(\mathbb{R}) \text{ for all } k \leq m\}.$$

Notice that $L^2_{\rm per}(0,\ell)$ can be identified with $L^2(0,\ell)$, but this is not true for the spaces H^m, $m \geq 1$. The results above, in particular the imbedding theorems, hold for these spaces, as well.

B Center Manifolds

The references in this section are to theorems, hypotheses, formulas, and remarks in Chapter 2.

B.1 Proof of Theorem 2.9 (Center Manifolds)

Consider system (2.1), and assume that Hypotheses 2.1, 2.4, and 2.7 hold. For any $u \in \mathscr{X}$ we set

$$u = u_0 + u_h \in \mathscr{X}, \quad u_0 = \mathbf{P}_0 u \in \mathscr{E}_0, \quad u_h = \mathbf{P}_h u \in \mathscr{X}_h,$$

and rewrite the system (2.1) as

$$\begin{aligned} \frac{du_0}{dt} - \mathbf{L}_0 u_0 &= \mathbf{P}_0 \mathbf{R}(u) \\ \frac{du_h}{dt} - \mathbf{L}_h u_h &= \mathbf{P}_h \mathbf{R}(u). \end{aligned} \tag{B.1}$$

Modified System

We take a cut-off function $\chi : \mathscr{E}_0 \to \mathbb{R}$ of class $\mathscr{C}^\infty$ such that

$$\chi(u_0) = \begin{cases} 1 \text{ for } \|u_0\| \leq 1 \\ 0 \text{ for } \|u_0\| \geq 2 \end{cases}, \qquad \chi(u_0) \in [0,1] \text{ for all } u_0 \in \mathscr{E}_0.$$

Since $\mathscr{E}_0$ is finite-dimensional such a function always exists. We use this function to modify the nonlinear terms $\mathbf{R}(u)$ outside a neighborhood of the origin, in order to be able to control the norm of the u_0-component of the system (B.1) in the space of exponentially growing functions $\mathscr{C}_\eta(\mathbb{R}, \mathscr{E}_0)$.

We set

$$\mathbf{R}^\varepsilon(u) = \chi\left(\frac{u_0}{\varepsilon}\right) \mathbf{R}(u) \text{ for all } \varepsilon \in (0, \varepsilon_0),$$

where ε_0 is chosen such that

$$\left\{u = u_0 + u_h \, ; \, \|u_0\|_{\mathscr{E}_0} \leq 2\varepsilon_0, \, \|u_h\|_{\mathscr{X}_h} \leq \varepsilon_0\right\} \subset \mathscr{V},$$

with $\mathscr{V}$ the neighborhood of the origin in Hypothesis 2.1. Then $\mathbf{R}^\varepsilon$ is well defined in the closed set

$$\mathscr{O}_\varepsilon = \mathscr{E}_0 \times B_\varepsilon(\mathscr{X}_h), \quad B_\varepsilon(\mathscr{X}_h) = \{u_h \in \mathscr{X}_h \, ; \, \|u_h\| \leq \varepsilon\},$$

and satisfies

$$\mathbf{R}^\varepsilon(u) = \mathbf{R}(u) \text{ for all } u \in \mathscr{O}_\varepsilon, \, \|u_0\| \leq \varepsilon.$$

Consider the modified system

$$\begin{aligned}\frac{du_0}{dt} - \mathbf{L}_0 u_0 &= \mathbf{P}_0 \mathbf{R}^\varepsilon(u)\\ \frac{du_h}{dt} - \mathbf{L}_h u_h &= \mathbf{P}_h \mathbf{R}^\varepsilon(u).\end{aligned} \tag{B.2}$$

The nonlinear terms in this system now satisfy

$$\begin{aligned}\delta_0(\varepsilon) &\stackrel{\text{def}}{=} \sup_{u\in\mathscr{O}_\varepsilon} \left(\|\mathbf{P}_0\mathbf{R}^\varepsilon(u)\|_{\mathscr{E}_0}, \|\mathbf{P}_h\mathbf{R}^\varepsilon(u)\|_{\mathscr{Y}_h}\right) = O(\varepsilon^2)\\ \delta_1(\varepsilon) &\stackrel{\text{def}}{=} \sup_{u\in\mathscr{O}_\varepsilon} \left(\|D_u\mathbf{P}_0\mathbf{R}^\varepsilon(u)\|_{\mathscr{L}(\mathscr{Z},\mathscr{E}_0)}, \|D_u\mathbf{P}_h\mathbf{R}^\varepsilon(u)\|_{\mathscr{L}(\mathscr{Z},\mathscr{Y}_h)}\right) = O(\varepsilon).\end{aligned} \tag{B.3}$$

We prove below the existence of a "global" center manifold for this system which, due to the fact that $\mathbf{R}^\varepsilon$ and $\mathbf{R}$ coincide for $\|u_0\|_{\mathscr{E}_0} \leq \varepsilon$, will give the local center manifold for the system (2.1) in the theorem.

Integral Formulation

We replace system (B.2) by the integral formulation

$$\begin{aligned}u_0(t) &= \mathbf{S}_{0,\varepsilon}(u,t,u_0(0)) \stackrel{\text{def}}{=} e^{\mathbf{L}_0 t}u_0(0) + \int_0^t e^{\mathbf{L}_0(t-s)}\mathbf{P}_0\mathbf{R}^\varepsilon(u(s))ds\\ u_h &= \mathbf{S}_{h,\varepsilon}(u) \stackrel{\text{def}}{=} \mathbf{K}_h\mathbf{P}_h\mathbf{R}^\varepsilon(u).\end{aligned} \tag{B.4}$$

The first equation in this system is obtained by the variation of constant formula from the first equation in (B.1). Here $u_0(0) \in \mathscr{E}_0$ is arbitrary, and the exponential $e^{\mathbf{L}_0 t}$ exists since $\mathscr{E}_0$ is finite-dimensional. The second equation in (B.4) is obtained from Hypothesis 2.7, used with $f \in \mathscr{C}_0(\mathbb{R},\mathscr{Y}_h)$. It is now straightforward to check that this integral system is equivalent to (B.2) for

$$u = (u_0,u_h) \in \mathscr{N}_{\eta,\varepsilon} \stackrel{\text{def}}{=} \mathscr{C}_\eta(\mathbb{R},\mathscr{E}_0) \times \mathscr{C}_0(\mathbb{R},B_\varepsilon(\mathscr{Z}_h)),$$

with $0 < \eta \leq \gamma$ and $\varepsilon \in (0,\varepsilon_0)$. Notice that $\mathscr{N}_{\eta,\varepsilon}$ is a closed subspace of $\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$, so that it is complete when equipped with the norm of $\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$.

Fixed Point Argument

Our aim now is to show that (B.4) has a unique solution $u = (u_0,u_h) \in \mathscr{N}_{\eta,\varepsilon}$ for any $u_0(0) \in \mathscr{E}_0$. For this we use a fixed point argument for the map

$$\mathbf{S}_\varepsilon(u,u_0(0)) \stackrel{\text{def}}{=} (\mathbf{S}_{0,\varepsilon}(u,\cdot,u_0(0)),\mathbf{S}_{h,\varepsilon}(u)), \quad \mathbf{S}_\varepsilon(\cdot,u_0(0)) : \mathscr{N}_{\eta,\varepsilon} \to \mathscr{N}_{\eta,\varepsilon}.$$

We show that $\mathbf{S}_\varepsilon(\cdot,u_0(0))$ is well defined and that it is a contraction with respect to the norm of $\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$ for $\eta\in(0,\gamma]$, with γ the constant in Hypothesis 2.7, and ε sufficiently small.

First, Hypothesis 2.4 implies that for any $\delta>0$ there is a constant $c_\delta>0$ such that

$$\|e^{\mathbf{L}_0 t}\|_{\mathscr{L}(\mathscr{E}_0)}\leq c_\delta e^{\delta|t|}\text{ for all }t\in\mathbb{R}. \tag{B.5}$$

Using this equality with $\delta=\eta$, we find

$$\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\left\|e^{\mathbf{L}_0 t}u_0(0)\right\|_{\mathscr{E}_0}\right)\leq c_\eta\|u_0(0)\|_{\mathscr{E}_0},$$

which shows that the first term in $\mathbf{S}_{0,\varepsilon}(u,\cdot,u_0(0))$ belongs to $\mathscr{C}_\eta(\mathbb{R},\mathscr{E}_0)$, for any $\eta>0$. Next, for any $u\in\mathscr{N}_{\eta,\varepsilon}$, we have the estimates

$$\|\mathbf{P}_0\mathbf{R}^\varepsilon(u(t))\|_{\mathscr{E}_0}\leq\delta_0(\varepsilon),\quad\|\mathbf{P}_h\mathbf{R}^\varepsilon(u(t))\|_{\mathscr{Y}_h}\leq\delta_0(\varepsilon),$$

which together with (B.5) for $\delta=\eta/2$, and Hypothesis 2.7 imply

$$\begin{aligned}\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\left\|\int_0^t e^{\mathbf{L}_0(t-s)}\mathbf{P}_0\mathbf{R}^\varepsilon(u(s))ds\right\|_{\mathscr{E}_0}\right)&\leq c_\delta\delta_0(\varepsilon)\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\int_0^t e^{\delta|t-s|}ds\right)\\&\leq\frac{2c_{\eta/2}\delta_0(\varepsilon)}{\eta},\end{aligned}$$

and

$$\|\mathbf{K}_h\mathbf{P}_h\mathbf{R}^\varepsilon(u)\|_{\mathscr{C}_0(\mathbb{R},\mathscr{Z}_h)}\leq C(0)\delta_0(\varepsilon).$$

This shows that $\mathbf{S}_\varepsilon(u,u_0(0))\in\mathscr{N}_{\eta,\varepsilon}$, provided $C(0)\delta_0(\varepsilon)\leq\varepsilon$, which holds for ε sufficiently small since $\delta_0(\varepsilon)=O(\varepsilon^2)$.

Now we show that the map $\mathbf{S}_\varepsilon(\cdot,u_0(0))$ is a contraction with respect to the norm of $\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$ for $\eta\in(0,\gamma]$ and sufficiently small ε. From equality (B.3) we find that

$$\begin{aligned}\|\mathbf{R}^\varepsilon(u_1)-\mathbf{R}^\varepsilon(u_2)\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Y})}&=\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\|\mathbf{R}^\varepsilon(u_1(t))-\mathbf{R}^\varepsilon(u_2(t))\|_{\mathscr{Y}}\right)\\&\leq\delta_1(\varepsilon)\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\|u_1(t)-u_2(t)\|_{\mathscr{Z}}\right)\\&\leq\delta_1(\varepsilon)\|u_1-u_2\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})}\end{aligned}$$

for any $u_1,u_2\in\mathscr{N}_{\eta,\varepsilon}$. Now, using (B.5) with $\delta=\eta/2$ we obtain

$$\begin{aligned}&\|\mathbf{S}_{0,\varepsilon}(u_1,\cdot,u_0(0))-\mathbf{S}_{0,\varepsilon}(u_2,\cdot,u_0(0))\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{E}_0)}\\&\qquad\leq c_\delta\delta_1(\varepsilon)\sup_{t\in\mathbb{R}}\left(e^{-\eta|t|}\left|\int_0^t e^{\eta|s|+\delta|t-s|}ds\right|\right)\|u_1-u_2\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})}\\&\qquad\leq\frac{2c_{\eta/2}\delta_1(\varepsilon)}{\eta}\|u_1-u_2\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})},\end{aligned}$$

and using the estimate in Hypothesis 2.7 we find

$$\|\mathbf{S}_{h,\varepsilon}(u_1) - \mathbf{S}_{h,\varepsilon}(u_2)\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_h)} \leq C(\eta)\delta_1(\varepsilon)\|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})}.$$

Since $\delta_1(\varepsilon) = O(\varepsilon)$ for any $\eta \in (0,\gamma]$, we can choose ε small enough such that

$$\|\mathbf{S}_\varepsilon(u_1,u_0(0)) - \mathbf{S}_\varepsilon(u_2,u_0(0))\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})} \leq \frac{1}{2}\|u_1 - u_2\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{Z})}.$$

Consequently, the map $\mathbf{S}_\varepsilon(\cdot,u_0(0))$ is a contraction in the complete metric space $\mathscr{N}_{\eta,\varepsilon}$.

Applying the fixed point theorem we now have the existence of a unique solution of (B.4),

$$u \stackrel{\text{def}}{=} \boldsymbol{\Phi}(u_0(0)) \in \mathscr{N}_{\eta,\varepsilon}$$

for any $u_0(0) \in \mathscr{E}_0$, for any $\eta \in (0,\gamma]$, and ε sufficiently small. Clearly, this is also a solution of (B.2).

Properties of $\boldsymbol{\Phi}$

Recall that ε is chosen such that

$$C(0)\delta_0(\varepsilon) \leq \varepsilon, \quad \frac{2c_{\eta/2}\delta_1(\varepsilon)}{\eta} \leq \frac{1}{2}, \quad C(\eta)\delta_1(\varepsilon) \leq \frac{1}{2}.$$

Then the continuity on $[0,\gamma]$ of the map $\eta \to C(\eta)$ in Hypothesis 2.7 implies that for any $\widetilde{\eta} \in (0,\gamma)$, we can choose $\varepsilon > 0$ such that these inequalities hold for all $\eta \in [\widetilde{\eta},\gamma]$. Consequently, for any $\widetilde{\eta} \in (0,\gamma)$, there exists $\varepsilon > 0$ such that the unique fixed point $\boldsymbol{\Phi}(u_0(0))$ belongs to $\mathscr{N}_{\eta,\varepsilon}$ for any $\eta \in [\widetilde{\eta},\gamma]$. This property is used later when showing that the center manifold is of class $\mathscr{C}^k$.

Next, notice that the map $u_0(0) \mapsto \mathbf{S}_{0,\varepsilon}(u,\cdot,u_0(0))$ is Lipschitz from $\mathscr{E}_0$ into $\mathscr{C}_\eta(\mathbb{R},\mathscr{E}_0)$, so that the map $u_0(0) \mapsto \mathbf{S}_\varepsilon(u,u_0(0))$ is also Lipschitz. Consequently, $\boldsymbol{\Phi}$ is a Lipschitz map. In addition, the uniqueness of the fixed point implies that

$$\boldsymbol{\Phi}(0) = 0.$$

Construction of $\boldsymbol{\Psi}$

We define now the map $\boldsymbol{\Psi} : \mathscr{E}_0 \to \mathscr{Z}_h$ in the theorem, through

$$(u_0(0), \boldsymbol{\Psi}(u_0(0))) \stackrel{\text{def}}{=} \boldsymbol{\Phi}(u_0(0))(0) \text{ for all } u_0(0) \in \mathscr{E}_0,$$

i.e., by taking the component in $\mathscr{Z}_h$ of the fixed point $\boldsymbol{\Phi}(u_0(0))$ at $t = 0$. Since $\boldsymbol{\Phi}$ is a Lipschitz map, we have that $\boldsymbol{\Psi}$ is also a Lipschitz map, and since $\boldsymbol{\Phi}(0) = 0$, we have

$$\Psi(0) = 0.$$

We prove now that Ψ has the properties (i) and (ii) in the theorem.

First, we show that the manifold

$$\mathscr{M}_{\eta,\varepsilon} = \{(u_0, \Psi(u_0))\ ;\ u_0 \in \mathscr{E}_0\}$$

is a global invariant manifold for the flow defined by (B.2). We define the shift operator Γ_s through

$$(\Gamma_s u)(t) = u(t+s) \text{ for all } t,\ s \in \mathbb{R}.$$

Since system (B.2) is autonomous, it is equivariant under the action Γ_s for any $s \in \mathbb{R}$, so that if u is a solution of (B.2), then $\Gamma_s u$ is also a solution of (B.2). Moreover, $\Gamma_s u \in \mathscr{N}_{\eta,\varepsilon}$ when $u \in \mathscr{N}_{\eta,\varepsilon}$.

Consider a solution u of (B.2) with $u(0) = (u_0(0), \Psi(u_0(0)))$ for some $u_0(0) \in \mathscr{E}_0$. Then $u = \Phi(u_0(0)) \in \mathscr{N}_{\eta,\varepsilon}$, and since $\Gamma_s u \in \mathscr{N}_{\eta,\varepsilon}$ is also a solution, from the uniqueness of the fixed point we conclude that

$$\Gamma_s u = \Phi(u_0(s)) \text{ for all } s \in \mathbb{R}.$$

Consequently,

$$u(s) = (u_0(s), \Psi(u_0(s)) \text{ for all } s \in \mathbb{R},$$

which shows that $\mathscr{M}_{\eta,\varepsilon}$ is globally invariant under the flow defined by (B.2). Since the system (B.1) coincides with (B.2) in

$$\mathscr{O}_\varepsilon = B_\varepsilon(\mathscr{E}_0) \times B_\varepsilon(\mathscr{Z}_h),$$

this proves part (i) of the theorem with $\mathscr{M}_0 = \mathscr{M}_{\eta,\varepsilon}$ and $\mathscr{O} = \mathscr{O}_\varepsilon$. Indeed, assume that u is a solution of (B.1) such that $u(0) \in \mathscr{M}_0 \cap \mathscr{O}$ and $u(t) \in \mathscr{O}$ for all $t \in [0,T]$. Then u satisfies (B.2) for all $t \in [0,T]$, and since $u(0) \in \mathscr{M}_{\eta,\varepsilon}$ and $\mathscr{M}_{\eta,\varepsilon}$ is an invariant manifold, we have $u(t) \in \mathscr{M}_{\eta,\varepsilon} = \mathscr{M}_0$ for all $t \in [0,T]$.

Consider now a solution u of (B.1) which belongs to $\mathscr{O} = \mathscr{O}_\varepsilon$ for all $t \in \mathbb{R}$. Then $u \in \mathscr{N}_{\eta,\varepsilon}$ and it is also a solution of (B.2). Consequently, $u = \Phi(u_0(0))$, so that $u(0) \in \mathscr{M}_{\eta,\varepsilon} = \mathscr{M}_0$, which proves part (ii) of the theorem.

Regularity of Ψ

We have proved so far that Ψ is a Lipschitz map. Notice that for this proof we have only used the fact that $\mathbf{R}$ is of class $\mathscr{C}^1$. It remains to show that Ψ is of class $\mathscr{C}^k$ when $\mathbf{R}$ is of class $\mathscr{C}^k$. For this, it is enough to prove that Φ is of class $\mathscr{C}^k$.

The major difficulty in proving this property comes from the fact that the *Nemitsky operator*

$$\mathbf{R}^\varepsilon : \mathscr{C}_\eta(\mathbb{R}, \mathscr{Z}) \to \mathscr{C}_\eta(\mathbb{R}, \mathscr{Y})$$

is not continuously differentiable, due to the growth of $u \in \mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$ as $t \to \pm\infty$. The following properties of this operator are proved in [120, Lemma 3.7]:

(i) $\mathbf{R}^\varepsilon : \mathscr{C}_\eta(\mathbb{R},\mathscr{Z}) \to \mathscr{C}_\zeta(\mathbb{R},\mathscr{Y})$ is continuous for any $\eta \geq 0$ and $\zeta > 0$;
(ii) $\mathbf{R}^\varepsilon : \mathscr{C}_\eta(\mathbb{R},\mathscr{Z}) \to \mathscr{C}_\zeta(\mathbb{R},\mathscr{Y})$ is of class $\mathscr{C}^k$ for any $0 \leq \eta < \zeta/k$ and $\zeta > 0$.

We point out that the kth order derivative exists for $\eta = \zeta/k$, but this derivative is continuous only if $\eta < \zeta/k$.

Following [120], the integral system (B.4) is written as

$$u = \mathbf{S}u_0(0) + \mathbf{K}\mathbf{R}^\varepsilon(u), \tag{B.6}$$

with $\mathbf{S}$ and $\mathbf{K}$ linear maps defined by

$$(\mathbf{S}u_0(0))(t) = e^{\mathbf{L}_0 t}u_0(0),$$

and

$$(\mathbf{K}v)(t) = \int_0^t e^{\mathbf{L}_0(t-s)}\mathbf{P}_0(v(s))ds + (\mathbf{K}_h\mathbf{P}_h(v))(t).$$

We already showed that

$$\mathbf{S} \in \mathscr{L}(\mathscr{E}_0, \mathscr{C}_{\widetilde{\eta}}(\mathbb{R},\mathscr{E}_0)), \quad \|\mathbf{S}u_0(0)\|_{\mathscr{C}_\eta(\mathbb{R},\mathscr{E}_0)} \leq c_{\eta/2}\|u_0(0)\|_{\mathscr{E}_0},$$

and that $\mathbf{K}\mathbf{R}^\varepsilon : \mathscr{N}_{\eta,\varepsilon} \to \mathscr{N}_{\eta,\varepsilon}$ is a contraction for any $\eta \in [\widetilde{\eta},\gamma]$, when $\widetilde{\eta} \in (0,\gamma)$ and ε is sufficiently small.

The idea is to consider the fixed point $u = \boldsymbol{\Phi}(u_0(0)) \in \mathscr{N}_{\eta,\varepsilon} \subset \mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$ of (B.4) found for $\eta \in [\widetilde{\eta},\gamma]$, with $\widetilde{\eta}$ taken such that $0 < \widetilde{\eta} < \gamma/k$, and to show that the map $\boldsymbol{\Phi} : \mathscr{E}_0 \to \mathscr{C}_\eta(\mathbb{R},\mathscr{Z})$ is of class $\mathscr{C}^k$ for all $\eta \in (k\widetilde{\eta},\gamma]$, with

$$D^p\boldsymbol{\Phi}(u_0(0)) \in \mathscr{L}^p(\mathscr{E}_0, \mathscr{C}_{k\widetilde{\eta}}(\mathbb{R},\mathscr{Z})).$$

Here $\mathscr{L}^p(\mathscr{E}_0, \mathscr{C}_{k\widetilde{\eta}}(\mathbb{R},\mathscr{Z}))$ denotes the Banach space of p-linear continuous maps from $\mathscr{E}_0$ into $\mathscr{C}_{k\widetilde{\eta}}(\mathbb{R},\mathscr{Z})$. Several proofs of this result are available in the literature, all being quite long and technical. In particular, in [121] an abstract theorem for contractions on embedded Banach spaces is used, whereas in [120] a fiber contraction theorem due to Hirsch and Pugh [48] is used. While we refer to these works for further details, we only point out that the derivative $D\boldsymbol{\Phi}(u_0(0))$ is the fixed point in $\mathscr{L}(\mathscr{E}_0, \mathscr{C}_{\widetilde{\eta}}(\mathbb{R},\mathscr{Z}))$ of the linear equation

$$D\boldsymbol{\Phi}(u_0(0)) = \mathbf{S} + \mathbf{K}D_u\mathbf{R}^\varepsilon(\boldsymbol{\Phi}(u_0(0))D\boldsymbol{\Phi}(u_0(0)),$$

which may be differentiated up to order k. In particular, this implies that $D\mathbf{P}_h\boldsymbol{\Phi}(0) = 0$ and $D\boldsymbol{\Psi}(0) = 0$, and ends the proof of Theorem 2.9. $\square$

B.2 Proof of Theorem 2.17 (Semilinear Case)

We prove here the first part of Theorem 2.17. The second part is an immediate consequence of this and of Theorem 2.9.

Estimates on the Resolvent of $\mathbf{L}_h$

First, the estimates on the resolvent (2.9) and (2.10) together with the fact that $\mathbf{L}_h$ has no spectrum on the imaginary axis, i.e., $i\omega\mathbb{I}-\mathbf{L}_h$ is invertible for any $\omega\in\mathbb{R}$, imply that there exists a positive constant c_1 such that for any $\omega\in\mathbb{R}$, the following estimates hold:

$$\|(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{X}_h)}\leq\frac{c_1}{1+|\omega|},\quad \|(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{Z}_h)}\leq\frac{c_1}{1+|\omega|},\tag{B.7}$$

$$\|(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{X}_h,\mathscr{Z}_h)}\leq c_1,\quad \|(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{Y}_h,\mathscr{Z}_h)}\leq\frac{c_1}{(1+|\omega|)^{1-\alpha}}.\tag{B.8}$$

Next, we claim that there exist $\delta>0$ and $M>0$ such that any $\lambda\in\mathbb{C}$ satisfying

$$\lambda=\mu+i\omega,\quad |\mu|\leq\delta(1+|\omega|)$$

belongs to the resolvent set of $\mathbf{L}_h$, and that the following estimates hold:

$$\|(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{X}_h)}\leq\frac{M}{1+|\lambda|},\quad \|(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{X}_h,\mathscr{Z}_h)}\leq M,\tag{B.9}$$

$$\|(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{Y}_h,\mathscr{Z}_h)}\leq\frac{M}{(1+|\lambda|)^{1-\alpha}}.\tag{B.10}$$

Indeed, we can write

$$\lambda\mathbb{I}-\mathbf{L}_h=\left(\mathbb{I}+\mu(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\right)(i\omega\mathbb{I}-\mathbf{L}_h)=(i\omega\mathbb{I}-\mathbf{L}_h)\left(\mathbb{I}+\mu(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\right),$$

and choosing $\delta=1/2c_1$ from equalities (B.7) we find

$$\|\mu(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{X}_h)}\leq\frac{1}{2},\quad \|\mu(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}\|_{\mathscr{L}(\mathscr{Z}_h)}\leq\frac{1}{2}.$$

This shows that the operator $\mathbb{I}+\mu(i\omega\mathbb{I}-\mathbf{L}_h)^{-1}$ has a bounded inverse in both $\mathscr{L}(\mathscr{X}_h)$ and $\mathscr{L}(\mathscr{Z}_h)$, so that $\lambda\mathbb{I}-\mathbf{L}_h$ is invertible. Furthermore, inequalities (B.7) and (B.8) imply the inequalities above on the norms of $(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}$.

We set

$$\beta\stackrel{\text{def}}{=}\min\{|\operatorname{Re}\lambda|\,;\,\lambda\in\sigma(\mathbf{L}_h)\}\geq\delta>0.\tag{B.11}$$

(Recall that $\beta>\gamma>0$, according to Hypothesis 2.4(i).)

Construction of $\mathbf{S}_\pm$

Consider the curves Γ_+ and Γ_- in $\mathbb{C}$ defined by

$$\Gamma_+ = \{-\delta|\omega| + i\omega\,;\ \omega \in \mathbb{R}\}, \quad \Gamma_- = \{\delta|\omega| + i\omega\,;\ \omega \in \mathbb{R}\},$$

and oriented such that ω increases along Γ_+ and decreases along Γ_-. The results above imply that these two curves lie in the resolvent set of $\mathbf{L}_h$, and that for any λ on one of these two curves the estimates (B.9) and (B.10) hold.

For $t > 0$ we define

$$\mathbf{S}_+(t) = \frac{1}{2i\pi}\int_{\Gamma_+} e^{\lambda t}(\lambda\mathbb{I} - \mathbf{L}_h)^{-1}d\lambda \in \mathscr{L}(\mathscr{X}_h, \mathscr{Z}_h),$$

for which the estimates (B.9) and the dominated convergence theorem allows us to show that it is well defined and that the map $t \mapsto \mathbf{S}_+(t)$ is differentiable with

$$\frac{d^n\mathbf{S}_+(t)}{dt^n} = (\mathbf{L}_h)^n\,\mathbf{S}_+(t) \text{ for all } n \geq 1. \tag{B.12}$$

Similarly, for $t < 0$ we set

$$\mathbf{S}_-(t) = \frac{1}{2i\pi}\int_{\Gamma_-} e^{\lambda t}(\lambda\mathbb{I} - \mathbf{L}_h)^{-1}d\lambda \in \mathscr{L}(\mathscr{X}_h, \mathscr{Z}_h),$$

for which we have that

$$\frac{d^n\mathbf{S}_-(t)}{dt^n} = (\mathbf{L}_h)^n\,\mathbf{S}_-(t) \text{ for all } n \geq 1. \tag{B.13}$$

Furthermore, the commutativity property

$$\mathbf{L}_h(\lambda\mathbb{I} - \mathbf{L}_h)^{-1} = (\lambda\mathbb{I} - \mathbf{L}_h)^{-1}\mathbf{L}_h$$

implies that

$$\begin{aligned} \mathbf{L}_h\mathbf{S}_+(t)u &= \mathbf{S}_+(t)\mathbf{L}_h u \text{ for all } u \in \mathscr{Z}_h,\ t > 0,\\ \mathbf{L}_h\mathbf{S}_-(t)u &= \mathbf{S}_-(t)\mathbf{L}_h u \text{ for all } u \in \mathscr{Z}_h,\ t < 0 \end{aligned}$$

and using the estimate (B.10) we show that for any fixed $\beta' < \beta$ and for $0 < \gamma' < \beta'$, there exists $M' > 0$ such that the estimates

$$\begin{aligned} \|\mathbf{S}_+(t)\|_{\mathscr{L}(\mathscr{Y}_h, \mathscr{Z}_h)} &\leq M'(1 + t^{-\alpha})e^{-\gamma' t} \text{ for all } t > 0,\\ \|\mathbf{S}_-(t)\|_{\mathscr{L}(\mathscr{Y}_h, \mathscr{Z}_h)} &\leq M'(1 + |t|^{-\alpha})e^{-\gamma'|t|} \text{ for all } t < 0, \end{aligned} \tag{B.14}$$

hold. The following lemma is proved at the end of this section.

Lemma B.1 *The limits*

$$\mathbf{P}_- = \lim_{t\to 0^+} \mathbf{S}_+(t)|_{\mathscr{Y}_h} \in \mathscr{L}(\mathscr{Y}_h, \mathscr{X}_h), \quad \mathbf{P}_+ = \lim_{t\to 0^-} \mathbf{S}_-(t)|_{\mathscr{Y}_h} \in \mathscr{L}(\mathscr{Y}_h, \mathscr{X}_h), \quad \text{(B.15)}$$

exist, and

$$(\mathbf{P}_+ + \mathbf{P}_-)u = u \text{ for all } u \in \mathscr{Y}_h. \quad \text{(B.16)}$$

Checking Hypothesis 2.7

We now use the operators $\mathbf{S}_+(t)$ and $\mathbf{S}_-(t)$ above to solve the linear differential equation

$$\frac{du_h}{dt} = \mathbf{L}_h u_h + f(t). \quad \text{(B.17)}$$

We show that for any $f \in \mathscr{C}_\eta(\mathbb{R}, \mathscr{Y}_h)$ with $\eta \in [0,\gamma]$ this equation has a unique solution $u_h \in \mathscr{C}_\eta(\mathbb{R}, \mathscr{Z}_h)$ given by

$$u_h(t) = (\mathbf{K}_h f)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{t} \mathbf{S}_+(t-s)f(s)ds - \int_{t}^{\infty} \mathbf{S}_-(t-s)f(s)ds, \quad \text{(B.18)}$$

with the properties in Hypothesis 2.7.

We assume that γ' in (B.14) is such that $\beta > \gamma' > \gamma$. Then using these two estimates and the dominated convergence theorem it is straightforward to check that $u_h = \mathbf{K}_h f \in \mathscr{C}_\eta(\mathbb{R}, \mathscr{Z}_h)$, and that the linear map $\mathbf{K}_h \in \mathscr{L}(\mathscr{C}_\eta(\mathbb{R}, \mathscr{Y}_h), \mathscr{C}_\eta(\mathbb{R}, \mathscr{Z}_h))$, with norm satisfying the inequality in Hypothesis 2.7. Moreover using (B.12), (B.13), and Lemma B.1, we obtain that in $\mathscr{X}_h$ the following holds

$$\begin{aligned}
\frac{du_h}{dt} &= \lim_{s\to t^-} \mathbf{S}_+(t-s)f(s) + \lim_{s\to t^+} \mathbf{S}_-(t-s)f(s) \\
&\quad + \int_{-\infty}^{t} \mathbf{L}_h \mathbf{S}_+(t-s)f(s)ds - \int_{t}^{\infty} \mathbf{L}_h \mathbf{S}_-(t-s)f(s)ds \\
&= \mathbf{L}_h u_h(t) + (\mathbf{P}_+ + \mathbf{P}_-)f(t) \\
&= \mathbf{L}_h u_h(t) + f(t).
\end{aligned}$$

Consequently, u_h is a solution of equation (B.17). It remains to prove the uniqueness of this solution.

Assume that $\widetilde{u}_h(t) \in \mathscr{C}_\eta(\mathbb{R}, \mathscr{Z}_h)$ is a solution of the homogeneous equation

$$\frac{d\widetilde{u}_h}{dt} = \mathbf{L}_h \widetilde{u}_h.$$

We show that $\widetilde{u}_h = 0$. Take any $t_0 \in \mathbb{R}$, and define

$$\begin{aligned}
\widetilde{u}_+(t) &\stackrel{\text{def}}{=} \mathbf{S}_+(t_0 - t)\widetilde{u}_h(t) \text{ for all } t < t_0, \\
\widetilde{u}_-(t) &\stackrel{\text{def}}{=} \mathbf{S}_-(t_0 - t)\widetilde{u}_h(t) \text{ for all } t > t_0.
\end{aligned}$$

Then we have

$$\frac{d\widetilde{u}_+(t)}{dt} = -\mathbf{S}_+(t_0-t)\mathbf{L}_h\widetilde{u}_h(t) + \mathbf{S}_+(t_0-t)\frac{d\widetilde{u}_h}{dt}(t) = 0$$

in $\mathscr{X}_h$ for all $t < t_0$, hence

$$\widetilde{u}_+(t) = \lim_{s\to-\infty} \widetilde{u}_+(s) \text{ for all } t < t_0.$$

Using (B.21) and the continuous embedding from $\mathscr{Y}_h$ into $\mathscr{X}_h$, it follows that for $\eta < \gamma' < \beta$, there is a constant $C_{\gamma'}$ such that

$$\|\mathbf{S}_+(t)\|_{\mathscr{L}(\mathscr{Z}_h,\mathscr{X}_h)} \le C_{\gamma'} e^{-\gamma' t} \text{ for all } t > 0.$$

Consequently,

$$\|\widetilde{u}_+(s)\|_{\mathscr{X}_h} \le \|\mathbf{S}_+(t_0-s)\|_{\mathscr{L}(\mathscr{Z}_h,\mathscr{X}_h)}\|\widetilde{u}_h(s)\|_{\mathscr{Z}_h} \le C_{\gamma'} e^{-\gamma'(t_0-s)}e^{\eta|s|}\|\widetilde{u}_h\|_{\mathscr{C}_\eta},$$

so that $\|\widetilde{u}_+(s)\|_{\mathscr{X}_h} \to 0$ as $s \to -\infty$. This implies that $\widetilde{u}_+(t) = 0$ for all $t < t_0$, and similarly we find that $\widetilde{u}_-(t) = 0$ for all $t > t_0$. From the definitions of $\mathbf{P}_+$ and $\mathbf{P}_-$, and from Lemma B.1, we conclude that $\widetilde{u}_h(t_0) = 0$. Since t_0 is arbitrary we have that $\widetilde{u}_h = 0$, which completes the proof of Theorem 2.17. □

Remark B.2 *In the particular case when $\sigma^+ = \varnothing$, we can define the bounded projection $\mathbf{P}_- = \mathbb{I} - \mathbf{P}_0 = \mathbf{P}_h$. The estimates (B.9) imply in this case that the linear operator $\mathbf{L}_- = \mathbf{L}_h$ is the infinitesimal generator of an analytic semigroup $(\mathbf{S}_+(t))_{t\ge 0}$ in $\mathscr{X}_h$, which satisfies*

$$\|\mathbf{S}_+(t)\|_{\mathscr{L}(\mathscr{Z}_h)} \le ce^{-\gamma' t} \textit{ for all } t \ge 0.$$

This allows us to give a simpler proof of Theorem 2.17 in this case.

Proof (of Lemma B.1) For any $\eta > 0$ we define the paths Γ_+^η and Γ_-^η in $\mathbb{C}$ by

$$\Gamma_+^\eta = \{-\delta|\omega| + i\omega\,;\ \omega\in\mathbb{R},\ |\omega| \ge \delta^{-1}\eta\} \cup \{-\eta + i\omega\,;\ \omega\in\mathbb{R},\ |\omega| \le \delta^{-1}\eta\},$$
$$\Gamma_-^\eta = \{\delta|\omega| + i\omega\,;\ \omega\in\mathbb{R},\ |\omega| \ge \delta^{-1}\eta\} \cup \{\eta + i\omega\,;\ \omega\in\mathbb{R},\ |\omega| \le \delta^{-1}\eta\},$$

and orient them such that ω increases along Γ_+^η, and decreases along Γ_-^η (see Figure B.1).

For any $\eta \in (0,\beta)$ we can rewrite $\mathbf{S}_\pm(t)$ as

$$\mathbf{S}_+(t) = \frac{1}{2i\pi}\int_{\Gamma_+^\eta} e^{\lambda t}(\lambda\mathbb{I} - \mathbf{L}_h)^{-1}d\lambda \text{ for all } t > 0,$$
$$\mathbf{S}_-(t) = \frac{1}{2i\pi}\int_{\Gamma_-^\eta} e^{\lambda t}(\lambda\mathbb{I} - \mathbf{L}_h)^{-1}d\lambda \text{ for all } t < 0. \tag{B.19}$$

Using the identity

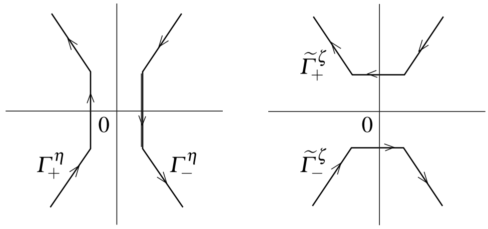

Fig. B.1 Plot in the complex plane of the paths $\Gamma_\pm^\eta$ (left) and $\widetilde{\Gamma}_\pm^\zeta$ (right).

$$\mathbf{L}_h(\lambda\mathbb{I}-\mathbf{L}_h)^{-1} = -\mathbb{I}_{\mathscr{X}_h} + \lambda(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}, \tag{B.20}$$

which holds for any $\lambda \in \rho(\mathbf{L}_h)$, we obtain

$$\mathbf{S}_+(t) = \frac{1}{2i\pi}\left(\int_{\Gamma_+^\eta} \frac{e^{\lambda t}}{\lambda}d\lambda\right)\mathbb{I}_{\mathscr{X}_h} + \frac{1}{2i\pi}\int_{\Gamma_+^\eta} \frac{e^{\lambda t}}{\lambda}\mathbf{L}_h(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda.$$

The first integral in the right hand side of this equality is independent of η, and by taking the limit as $\eta \to \infty$ we conclude that this integral vanishes.

Next, using (B.9) and the fact that $\alpha \in [0,1)$ we find that

$$\|\mathbf{S}_+(t)\|_{\mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)} \le C_\eta e^{-\eta t} \text{ for all } t>0, \tag{B.21}$$

for any $\eta \in (0,\beta)$, and we conclude that

$$\mathbf{P}_- = \lim_{t\to 0^+} \mathbf{S}_+(t)|_{\mathscr{Y}_h} = \frac{1}{2i\pi}\int_{\Gamma_+^\eta} \frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda \in \mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)$$

is well defined. Similarly, we find

$$\mathbf{P}_+ = \lim_{t\to 0^-} \mathbf{S}_-(t)|_{\mathscr{Y}_h} = \frac{1}{2i\pi}\int_{\Gamma_-^\eta} \frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda \in \mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)$$

for any $\eta \in (0,\beta)$, which proves the first part of the lemma.

In order to prove (B.16), we define for $\zeta > 0$ the paths $\widetilde{\Gamma}_\pm^\zeta$ by

$$\widetilde{\Gamma}_+^\zeta = \{\mu + i\delta^{-1}|\mu|\,;\, \mu\in\mathbb{R},\ |\mu|\ge\delta\zeta\} \cup \{\mu+i\zeta\,;\,\mu\in\mathbb{R},\ |\mu|\le\delta\zeta\},$$
$$\widetilde{\Gamma}_-^\zeta = \{\mu - i\delta^{-1}|\mu|\,;\, \mu\in\mathbb{R},\ |\mu|\ge\delta\zeta\} \cup \{\mu-i\zeta\,;\,\mu\in\mathbb{R},\ |\mu|\le\delta\zeta\},$$

oriented such that μ decreases along $\widetilde{\Gamma}_+^\zeta$, and increases along $\widetilde{\Gamma}_-^\zeta$. Now observe that the operators

$$\mathbf{B}_+ \stackrel{\text{def}}{=} \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_+^\zeta} \frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda,$$
$$\mathbf{B}_- \stackrel{\text{def}}{=} \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_-^\zeta} \frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda,$$

are independent of ζ, and that the dominated convergence theorem shows their limit in $\mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)$, as $\zeta\to\infty$, vanishes. Consequently, $\mathbf{B}_\pm = 0$.

Next, for $\eta = \delta\zeta$, we define the oriented clockwise rectangular path

$$\Gamma_\eta = \Gamma_+^\eta + \Gamma_-^\eta - \widetilde{\Gamma}_+^\zeta - \widetilde{\Gamma}_-^\zeta.$$

Then we have

$$\begin{aligned}\mathbf{P}_+ + \mathbf{P}_- &= \mathbf{B}_+ + \mathbf{B}_- + \frac{1}{2i\pi}\int_{\Gamma_\eta}\frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda \\ &= \frac{1}{2i\pi}\int_{\Gamma_\eta}\frac{\mathbf{L}_h}{\lambda}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda\end{aligned}$$

in $\mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)$, and by (B.20),

$$\mathbf{P}_+ + \mathbf{P}_- = \frac{1}{2i\pi}\int_{\Gamma_\eta}(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}d\lambda - \frac{1}{2i\pi}\left(\int_{\Gamma_\eta}\frac{d\lambda}{\lambda}\right)\mathbb{I}_{\mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)},$$

where $\mathbb{I}_{\mathscr{L}(\mathscr{Y}_h,\mathscr{X}_h)}$ denotes the continuous embedding from $\mathscr{Y}_h$ into $\mathscr{X}_h$. The first integral on the right hand side of this equality vanishes, since the interior of the rectangle Γ_η does not contain any singularity of $(\lambda\mathbb{I}-\mathbf{L}_h)^{-1}$, which then proves (B.16), and completes the proof of the lemma. □

B.3 Proof of Theorem 3.9 (Nonautonomous Vector Fields)

The proof of Theorem 3.9 in Chapter 2 follows exactly the proof of Theorem 2.9 given in Section B.1. The main difference is that we replace the arbitrary data $u_0(0)$ in the integral formulation (B.4) by $u_0(s) = v_0$, which modifies trivially (B.4). The fixed point is now denoted by $\boldsymbol{\Phi}(v_0,s)$, so that the corresponding solution of the system (3.9), modified by the cut-off function, is

$$u(v_0,s,t) = u_0(v_0,s,t) + (\boldsymbol{\Phi}(v_0,s))(t),$$

where $u_0(v_0,s,s) = v_0 \in \mathscr{E}_0$. We set

$$\boldsymbol{\Psi}(u_0,t) \stackrel{\text{def}}{=} (\boldsymbol{\Phi}(u_0,t))(t).$$

Then the uniqueness of the fixed point implies that

$$(\Phi(u_0(v_0,s,\tau),\tau))(t) = (\Phi(v_0,s))(t),$$

hence

$$\Psi(u_0(v_0,s,t),t) = (\Phi(v_0,s))(t).$$

This proves that

$$u_0(v_0,s,t) + \Psi(u_0(v_0,s,t),t) = u(v_0,s,t),$$

i.e., the set

$$\{(t,u_0 + \Psi(u_0,t))\ ;\ (u_0,t) \in \mathscr{E}_0 \times \mathbb{R}\}$$

is an integral manifold for (3.9) modified by the cut-off function. Restricting to the ball $B_\varepsilon(\mathscr{E}_0)$, this implies property (i) of the theorem. Property (ii) is obtained as for Theorem 2.9. □

We conclude this section with a brief proof of the particular cases in Corollary 3.11.

Property (i) results from a standard property of τ-periodic systems, which implies here that

$$u(v_0,s,t) = u(v_0,s+\tau,t+\tau).$$

This leads directly to

$$\Psi(u_0,t) = \Psi(u_0,t+\tau).$$

Part (ii) is obtained from the integral formulations (B.4) for both $u(v_0,s,t)$ and $u_\infty(v_0,s,t)$ by estimating, in a straightforward way, their difference as $t \to \infty$. □

B.4 Proof of Theorem 3.13 (Equivariant Systems)

The uniqueness of the global center manifold for the modified system (B.2) in Section B.1 implies that this manifold is invariant under $\mathbf{T}$, provided system (B.2) is equivariant under $\mathbf{T}$. Since equation (2.1) is equivariant under $\mathbf{T}$, the modified is also equivariant when the cut-off function χ satisfies

$$\chi(\mathbf{T}_0 u_0) = \chi(u_0) \text{ for all } u_0 \in \mathscr{E}_0. \tag{B.22}$$

Taking the Euclidean norm in $\mathscr{E}_0$, which is finite-dimensional, for any isometry $\mathbf{T}_0$ on $\mathscr{E}_0$ we can choose χ to be a smooth function of $\|u_0\|^2$, such that (B.22) holds. Consequently, the result in the theorem follows from the fact that $\mathbf{T}_0$ is an isometry on $\mathscr{E}_0$. □

B.5 Proof of Theorem 3.22 (Empty Unstable Spectrum)

With the notations from Section B.1, assume that $u(\cdot;u(0)) \in \mathscr{C}^0(\mathbb{R}^+,\mathscr{Z})$ is a solution of (2.1), which belongs to $\mathscr{O}_{\varepsilon/4}$ for all $t \geq 0$. Consider $\overline{u}(t,u(0))$ defined through

$$\overline{u}(t;u(0)) = \begin{cases} u(t;u(0)) \text{ for } t \geq 0 \\ \widetilde{u}(t;u(0)) \text{ for } t \leq 0, \end{cases}$$

where $\widetilde{u}(\cdot;u(0))$ is the solution of the modified equation

$$\frac{d\widetilde{u}}{dt} = \mathbf{L}_0\widetilde{u} + \mathbf{P}_0\mathbf{R}^\varepsilon(\widetilde{u}), \quad \widetilde{u}(0) = u(0).$$

Notice that $\mathbf{P}_-\widetilde{u}(t) = \mathbf{P}_-u(0)$, where $\mathbf{P}_- = \mathbb{I} - \mathbf{P}_0 = \mathbf{P}_h$. Then we find that

$$\sup_{t\leq 0}\left(e^{\eta t}\|\mathbf{P}_0\overline{u}(t;u(0))\|_{\mathscr{E}_0}\right) < \infty,$$

and since

$$\|\mathbf{P}_0 u(t;u(0))\|_{\mathscr{E}_0} \leq \frac{\varepsilon}{4}, \quad \|\mathbf{P}_- u(t;u(0))\|_{\mathscr{Z}} \leq \frac{\varepsilon}{4} \text{ for all } t \geq 0,$$

we have that $\overline{u}(\cdot;u(0) \in \mathscr{N}_{\eta,\varepsilon/4}$. Moreover, for all $t \in \mathbb{R}$,

$$\mathbf{P}_0\overline{u}(t;u(0)) = e^{\mathbf{L}_0 t}\mathbf{P}_0 u(0) + \int_0^t e^{\mathbf{L}_0(t-\tau)}\mathbf{P}_0\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))d\tau. \tag{B.23}$$

Now assume that we have found a solution

$$z \in \mathscr{F}_\eta(\mathbb{R},\mathscr{E}_0) \times \left(\mathscr{C}_0(\mathbb{R},B_{\varepsilon/2}(\mathscr{Z}_-)) \cap \mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_-)\right),$$

which means that $z(t) \to 0$ exponentially as $t \to +\infty$, of the equation

$$\begin{aligned} z(t) \;=\; & -\mathbf{P}_-\overline{u}(t;u(0)) + \mathbf{K}_h\mathbf{P}_-\mathbf{R}^\varepsilon(\overline{u}(\cdot;u(0)) + z)(t) \\ & - \int_t^\infty e^{\mathbf{L}_0(t-\tau)}\mathbf{P}_0\left[\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)) + z(\tau)) - \mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))\right]d\tau. \end{aligned} \tag{B.24}$$

We show below that $t \mapsto \overline{u}(t;u(0)) + z(t)$ is solution of (B.2), which belongs to $\mathscr{N}_{\eta,\varepsilon}$. As a consequence, $u(0) + z(0) \in \mathscr{M}_0$, and by construction

$$u(t,u(0) + z(0)) = u(t;u(0)) + z(t) \text{ for all } t > 0.$$

This is precisely the assertion in Theorem 3.22, since we have found a solution lying on $\mathscr{M}_0$ which is exponentially asymptotic to the solution $u(\cdot;u(0))$ of the initial value problem.

To end the proof, we show that $t \mapsto \overline{u}(t;u(0)) + z(t)$ is solution of (B.2). First, from (B.24) we observe that

$$\mathbf{P}_0 z(0) = -\int_0^\infty e^{-\mathbf{L}_0\tau}\mathbf{P}_0\left[\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0))+z(\tau)) - \mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))\right]d\tau,$$

and using (B.23) we obtain

$$\begin{aligned}\overline{u}(t;u(0))+z(t) &= e^{\mathbf{L}_0 t}P_0(u(0)+z(0)) + \left(\mathbf{K}_h\mathbf{P}_-\mathbf{R}^\varepsilon(\overline{u}(\cdot;u(0))+z)\right)(t)\\ &\quad + \int_0^t e^{\mathbf{L}_0(t-\tau)}\mathbf{P}_0\mathbf{R}^\varepsilon\left(\overline{u}(\tau;u(0))+z(\tau)\right)d\tau.\end{aligned}$$

This is equivalent to the fact that $\overline{u}(\cdot;u(0))+z\in\mathscr{N}_{\eta,\varepsilon}$ satisfies (B.2). It remains then to prove the existence of a solution

$$z\in\mathscr{F}_\eta(\mathbb{R},\mathscr{E}_0)\times\left(\mathscr{C}_0(\mathbb{R},B_{\varepsilon/2}(\mathscr{Z}_-))\cap\mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_-)\right)$$

of (B.24).

The argument is similar to the proof of the existence of the fixed point in the proof of the center manifold theorem in Section B.1. The main difference is that the space $\mathscr{C}_\eta$ is replaced by the space $\mathscr{F}_\eta$. Then for $z\in\mathscr{F}_\eta(\mathbb{R},\mathscr{E}_0)\times\left(\mathscr{C}_0(\mathbb{R},B_{\varepsilon/2}(\mathscr{Z}_-))\cap\mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_-)\right)$, examining all terms in (B.24), we find that:

(i) $\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0))+z(\tau))-\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))\in\mathscr{F}_\eta(\mathbb{R},\mathscr{Y})$, and

$$t\mapsto -\int_t^\infty e^{\mathbf{L}_0(t-\tau)}\mathbf{P}_0\left[\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0))+z(\tau))-\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))\right]d\tau\in\mathscr{F}_\eta(\mathbb{R},\mathscr{E}_0);$$

(ii) $\mathbf{K}_h\mathbf{P}_-\left(\mathbf{R}^\varepsilon(\overline{u}(\cdot;u(0))+z)-\mathbf{R}^\varepsilon(\overline{u}(\tau;u(0)))\right)\in\mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_-)\cap\mathscr{C}_0(\mathbb{R},B_{\varepsilon/8}(\mathscr{Z}_-))$, by construction and from Hypothesis 3.20, for ε sufficiently small;

(iii) $v\stackrel{\text{def}}{=}\mathbf{K}_h\mathbf{P}_-(\mathbf{R}^\varepsilon(\overline{u}(\cdot;u(0)))-\mathbf{P}_-\overline{u}(t;u(0))\in\mathscr{F}_\eta(\mathbb{R},\mathscr{Z}_-)\cap\mathscr{C}_0(\mathbb{R},B_{3\varepsilon/8}(\mathscr{Z}_-))$.

The last property follows from the fact that $v\in\mathscr{C}_\eta(\mathbb{R},\mathscr{Z}_-)$, and v is by construction a solution of

$$\frac{dv}{dt}=\mathbf{L}_-v$$

for $t\geq 0$, which by Hypothesis 3.20 implies the exponential convergence to 0 as $t\to\infty$, that $\mathbf{K}_h\mathbf{P}_-(\mathbf{R}^\varepsilon(\overline{u}(\cdot;u(0)))\in\mathscr{C}_0(\mathbb{R},B_{\varepsilon/8}(\mathscr{Z}_-))$, and that $\mathbf{P}_-\overline{u}(t;u(0))=\mathbf{P}_-u(0)\in B_{\varepsilon/4}(\mathscr{Z}_-)$ for $t<0$. This completes the proof of Theorem 3.22. □

C Normal Forms

The references in this section are to theorems, hypotheses, formulas, and remarks in Chapter 3.

C.1 Proof of Lemma 1.13 (0^3 Normal Form)

The proof below can be found in [25] (see also [20] for a different proof).

We set

$$\mathbf{N}(u) = (\Phi_1(A,B,C), \Phi_2(A,B,C), \Phi_3(A,B,C)), \quad u = (A,B,C),$$

where Φ_1, Φ_2, and Φ_3 are polynomials in (A,B,C). Then we have $\mathbf{L}^* u = (0,A,B)$ and the characterization (1.5) leads to

$$\begin{aligned} A\frac{\partial \Phi_1}{\partial B} + B\frac{\partial \Phi_1}{\partial C} &= 0 \\ A\frac{\partial \Phi_2}{\partial B} + B\frac{\partial \Phi_2}{\partial C} &= \Phi_1 \\ A\frac{\partial \Phi_3}{\partial B} + B\frac{\partial \Phi_3}{\partial C} &= \Phi_2. \end{aligned} \tag{C.1}$$

Since A and $B^2 - 2AC$ are first integrals of the linear vector field $\mathbf{L}^*$, we choose the new variables

$$\widetilde{A} = A, \quad \widetilde{B} = B^2 - 2AC, \quad \widetilde{C} = B,$$

where the change of variables is nonsingular as soon as $A \neq 0$, and define

$$\widetilde{\Phi}_j(\widetilde{A},\widetilde{B},\widetilde{C}) = \Phi_j(A,B,C), \quad j = 1,2,3.$$

Then the equations (C.1) become

$$\widetilde{A}\frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{C}} = 0, \quad \widetilde{A}\frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{C}} = \widetilde{\Phi}_1, \quad \widetilde{A}\frac{\partial \widetilde{\Phi}_3}{\partial \widetilde{C}} = \widetilde{\Phi}_2, \tag{C.2}$$

so that

$$\widetilde{\Phi}_1(\widetilde{A},\widetilde{B},\widetilde{C}) = \phi_1(\widetilde{A},\widetilde{B}).$$

We claim that ϕ_1 is a polynomial in its arguments. Indeed, by construction we have that

$$\Phi_1(A,B,C) = \Phi_1\left(\widetilde{A}, \widetilde{C}, \frac{\widetilde{C}^2 - \widetilde{B}}{2\widetilde{A}}\right),$$

which shows that ϕ_1 is a polynomial in $\widetilde{B}$ with rational coefficients in $\widetilde{A}$. Consequently, we can write

$$\Phi_1(A,B,C) = \sum_{k\in\mathbb{N}} f_k(\widetilde{A})\widetilde{B}^k = \sum_{i\in\mathbb{Z},\, k\in\mathbb{N}} f_{ik} A^i \widetilde{B}^k.$$

Assume that $\iota = \min\{i \in \mathbb{Z}\,;\, f_{ik} \neq 0 \text{ for some } k \in \mathbb{N}\} < 0$. Multiplying by $A^{-\iota}$ and setting $A = 0$ yields

$$\sum_{k\in\mathbb{N}} f_{\iota k} B^{2k} = 0,$$

which implies that $f_{\iota k} = 0$. This contradicts the assumption $\iota < 0$, so that $\iota \geq 0$. Consequently, ϕ_1 is a polynomial of $(\widetilde{A}, \widetilde{B})$, which proves the claim.

We write now

$$\phi_1(\widetilde{A}, \widetilde{B}) = \widetilde{A} P_1(\widetilde{A}, \widetilde{B}) + \psi_1(\widetilde{B}),$$

where P_1 and ψ_1 are polynomials, $\psi_1(\widetilde{B}) = \phi_1(0, \widetilde{B})$. Solving the second equation in (C.2) leads to

$$\widetilde{\Phi}_2(\widetilde{A}, \widetilde{B}, \widetilde{C}) = BP_1(\widetilde{A}, \widetilde{B}) + \frac{B}{\widetilde{A}} \psi_1(\widetilde{B}) + \frac{\phi_2(\widetilde{A}, \widetilde{B})}{\widetilde{A}},$$

where the same proof as above shows that ϕ_2 is a polynomial in its arguments. Multiplying this equality by $\widetilde{A}$ and setting $\widetilde{A} = 0$, we obtain

$$B\psi_1(B^2) + \phi_2(0, B^2) = 0 \text{ for all } B.$$

The two terms on the right hand side of this equality have different parity, so that $\psi_1(B^2) = \phi_2(0, B^2) = 0$, and then

$$\psi_1(\widetilde{B}) = 0, \quad \phi_2(0, \widetilde{B}) = 0 \text{ for all } \widetilde{B}.$$

Consequently, the polynomial $\phi_2(\widetilde{A}, \widetilde{B})$ is divisible by $\widetilde{A}$ and we can write

$$\begin{aligned}
\widetilde{\Phi}_1(\widetilde{A}, \widetilde{B}, \widetilde{C}) &= AP_1(\widetilde{A}, \widetilde{B}), \\
\widetilde{\Phi}_2(\widetilde{A}, \widetilde{B}, \widetilde{C}) &= BP_1(\widetilde{A}, \widetilde{B}) + AP_2(\widetilde{A}, \widetilde{B}) + \psi_2(\widetilde{B}),
\end{aligned}$$

where P_2 and ψ_2 are polynomials.

Finally, solving the last equation in (C.2) leads to

$$\widetilde{\Phi}_3(\widetilde{A}, \widetilde{B}, \widetilde{C}) = CP_1(\widetilde{A}, \widetilde{B}) + BP_2(\widetilde{A}, \widetilde{B}) + \frac{B}{A} \psi_2(\widetilde{B}) + \frac{\phi_3(\widetilde{A}, \widetilde{B})}{A},$$

and the same arguments as the ones above for ψ_1 and ϕ_2 apply here for ψ_2 and ϕ_3. Then we conclude that $\psi_2(\widetilde{B}) = 0$, and $\phi_3(\widetilde{A}, \widetilde{B})/A = P_3(\widetilde{A}, \widetilde{B})$, which is a polynomial. This completes the proof of Lemma 1.13. □

C.2 Proof of Lemma 1.17 ($(i\omega)^2$ Normal Form)

We define

$$\mathbf{N}(u) = (\Phi_1(A, B, \overline{A}, \overline{B}), \Phi_2(A, B, \overline{A}, \overline{B}), \overline{\Phi_1(A, B, \overline{A}, \overline{B})}, \overline{\Phi_1(A, B, \overline{A}, \overline{B})}),$$

for $u = (A, B, \overline{A}, \overline{B})$, and consider the differential operator

$$\mathscr{D}^* = -i\omega A \frac{\partial}{\partial A} + (A - i\omega B) \frac{\partial}{\partial B} + i\omega \overline{A} \frac{\partial}{\partial \overline{A}} + (\overline{A} + i\omega \overline{B}) \frac{\partial}{\partial \overline{B}}.$$

Then using the characterization (1.5) in its complex form (see Remark 1.5), we find the system

$$\mathscr{D}^*\Phi_1 = -i\omega\Phi_1, \quad \mathscr{D}^*\Phi_2 = -i\omega\Phi_2 + \Phi_1.$$

First, notice that

$$\mathscr{D}^*A = -i\omega A, \quad \mathscr{D}^*B = A - i\omega B,$$

and that the equation $\mathscr{D}^*u = 0$ has the following three independent first integrals:

$$u_1 = A\overline{A}, \quad u_2 = i(A\overline{B} - \overline{A}B), \quad u_3 = i\omega\frac{B}{A} + \ln A.$$

Since $\mathscr{D}^*(\Phi_1/A) = 0$, we have that Φ_1/A is a first integral of $\mathscr{D}^*u = 0$, as well. Consequently,

$$\Phi_1(A, B, \overline{A}, \overline{B}) = A\phi(u_1, u_2, u_3) \tag{C.3}$$

for some function ϕ.

We claim that ϕ is a polynomial in u_1, u_2, and that it is independent of u_3. Indeed, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial u_1} &= \frac{1}{A^2}\frac{\partial\Phi_1}{\partial\overline{A}} - \frac{B}{A^3}\frac{\partial\Phi_1}{\partial\overline{B}} \\
\frac{\partial\phi}{\partial u_2} &= \frac{-i}{A^2}\frac{\partial\Phi_1}{\partial\overline{B}} \\
\frac{\partial\phi}{\partial u_3} &= \frac{1}{i\omega}\frac{\partial\Phi_1}{\partial B} + \frac{\overline{A}}{i\omega A}\frac{\partial\Phi_1}{\partial\overline{B}}.
\end{aligned}$$

Assume that Φ_1 is of degree $n-1$. Then

$$\frac{\partial^n\phi}{\partial u_j^n} = 0, \quad j = 1, 2, 3,$$

so that

$$\frac{\partial^k\phi}{\partial u_1^\alpha \partial u_2^\beta \partial u_3^\gamma} = 0 \quad \text{for } \alpha + \beta + \gamma = k \geq 3n.$$

This shows that ϕ is a polynomial in u_1, u_2, u_3. Next, assume that ϕ depends upon u_3. Comparing the behavior of Φ_1 and $A\phi$ in the equality (C.3), as $A \to \infty$, we obtain a contradiction between the polynomial behavior in the left hand one side and the logarithmic behavior in the right hand side. Consequently, ϕ does not depend upon u_3 and we can write

$$\Phi_1(A, B, \overline{A}, \overline{B}) = AP(u_1, u_2),$$

where P is a polynomial in its arguments.

Finally, notice that $BP(u_1, u_2)$ is a particular solution of the equation

$$\mathscr{D}^*\Phi_2 = -i\omega\Phi_2 + \Phi_1.$$

Then proceeding as above for Φ_1, we obtain that

$$\Phi_2(A,B,\overline{A},\overline{B}) = BP(u_1,u_2) + AQ(u_1,u_2),$$

which ends the proof of Lemma 1.17. $\square$

C.3 Proof of Lemma 1.18 $(0^2(\mathrm{i}\omega)$ *Normal Form)*

We define

$$\mathbf{N}(u) = (\Phi_1(A,B,C,\overline{C}), \Phi_2(A,B,C,\overline{C}), \Phi_3(A,B,C,\overline{C}), \overline{\Phi_3}(A,B,C,\overline{C}))$$

for $u = (A,B,C,\overline{C})$, and consider the differential operator

$$\mathscr{D}^* = A\frac{\partial}{\partial B} - i\omega C\frac{\partial}{\partial C} + i\omega\overline{C}\frac{\partial}{\partial \overline{C}}.$$

Using characterization (1.5), we have to solve the system

$$\mathscr{D}^*\Phi_1 = 0, \quad \mathscr{D}^*\Phi_2 = \Phi_1, \quad \mathscr{D}^*\Phi_3 = -i\omega\Phi_3. \tag{C.4}$$

First, notice that

$$x = A, \quad y = |C|^2, \quad z = A\ln C + i\omega B$$

are three independent first integrals of the linear equation $\mathscr{D}^* u = 0$. Using the local diffeomorphism $(A,B,C,\overline{C}) \mapsto (x,y,z,B)$, with Jacobian determinant $-A$, it is easy to show that Φ_1 expressed in the new variables, $\widetilde{\Phi_1}(x,y,z,B)$, satisfies

$$A\frac{\partial \widetilde{\Phi_1}}{\partial B} = 0.$$

Consequently, there is a function ϕ, which is smooth, except at the origin, such that

$$\Phi_1(A,B,C,\overline{C}) = \phi(x,y,z). \tag{C.5}$$

We claim that ϕ is a polynomial in x, y, and that it is independent of z. Indeed, we have the equalities

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{\partial \Phi_1}{\partial A} + \frac{i\ln C}{\omega}\frac{\partial \Phi_1}{\partial B} \\
\frac{\partial \phi}{\partial y} &= \frac{1}{C}\frac{\partial \Phi_1}{\partial \overline{C}} \\
\frac{\partial \phi}{\partial z} &= \frac{1}{i\omega}\frac{\partial \Phi_1}{\partial B},
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial^n \phi}{\partial x^n} &= \frac{\partial^n \Phi_1}{\partial A^n} + \left(\frac{i \ln C}{\omega}\right)^n \frac{\partial^n \Phi_1}{\partial B^n} \\
\frac{\partial^n \phi}{\partial y^n} &= \frac{1}{C^n} \frac{\partial^n \Phi_1}{\partial \overline{C}^n} \\
\frac{\partial^n \phi}{\partial z^n} &= \frac{1}{(i\omega)^n} \frac{\partial^n \Phi_1}{\partial B^n}.
\end{aligned}$$

The right hand sides of these equalities vanish for n sufficiently large, which implies that ϕ is a polynomial in its arguments. Next, assume that ϕ depends upon z. Comparing the behavior of Φ_1 and ϕ in the equality (C.5), as $A \to \infty$, we obtain a contradiction between the polynomial behavior in the left hand one side and the logarithmic behavior on the right hand side. Consequently, ϕ does not depend upon z and we can write

$$\Phi_1(A,B,C,\overline{C}) = \phi_0(A, |C|^2),$$

where ϕ_0 is a polynomial in its arguments.

Next, the second equation in (C.4) leads to

$$\Phi_2(A,B,C,\overline{C}) = \frac{B}{A}\phi_0(A,|C|^2) + \phi_1(A,|C|^2,z),$$

and using the fact that $A\Phi_2$ and ϕ_0 are polynomials, it follows that $A\phi_1 = \psi(A,|C|^2)$, with ψ a polynomial satisfying

$$B\phi_0(0,|C|^2) = \psi(0,|C|^2).$$

This implies that $\phi_0(0,|C|^2) = 0$, and that ϕ_1 is a polynomial in A, and $|C|^2$. Summarizing, there are two polynomials P_0 and P_1 in A and $|C|^2$, such that

$$\begin{aligned}
\Phi_1(A,B,C,\overline{C}) &= AP_0(A,|C|^2), \\
\Phi_2(A,B,C,\overline{C}) &= BP_0(A,|C|^2) + P_1(A,|C|^2).
\end{aligned}$$

Finally, the equation for Φ_3 in (C.4) leads to

$$\mathscr{D}^*(\overline{C}\Phi_3) = 0.$$

Consequently,

$$\overline{C}\Phi_3(A,B,C,\overline{C}) = \phi_2(A,|C|^2),$$

where ϕ_2 is a polynomial such that $\phi_2(A,0) = 0$, and we conclude that there is a polynomial P_2 such that

$$\Phi_3(A,B,C,\overline{C}) = CP_2(A,|C|^2).$$

This completes the proof of Lemma 1.18. □

C.4 Proof of Lemma 1.19 $(0^20^2$ *Normal Form)*

We define

$$\mathbf{N}(u) = (\Phi_1(A,B,C,D), \Phi_2(A,B,C,D), \Phi_3(A,B,C,D), \Phi_4(A,B,C,D))$$

for $u = (A,B,C,D)$, and consider the differential operator

$$\mathscr{D}^* = A\frac{\partial}{\partial B} + C\frac{\partial}{\partial D}.$$

Using characterization (1.5), we obtain that

$$\mathscr{D}^*\Phi_1 = 0, \quad \mathscr{D}^*\Phi_2 = \Phi_1, \quad \mathscr{D}^*\Phi_3 = 0, \quad \mathscr{D}^*\Phi_4 = \Phi_3. \tag{C.6}$$

First, notice that

$$A, \quad C, \quad \widetilde{B} = BC - AD$$

are three independent first integrals of the equation $\mathscr{D}^*u = 0$. Then it is not difficult to show that there is a function ϕ_1 that is smooth, except at the origin, such that

$$\Phi_1(A,B,C,D) = \phi_1(A,C,\widetilde{B}).$$

Furthermore, we have the identities

$$\begin{aligned}
\frac{\partial^n \phi_1}{\partial A^n} &= \left(\frac{\partial}{\partial A} + \frac{D}{C}\frac{\partial}{\partial B}\right)^n \Phi_1 \\
\frac{\partial^n \phi_1}{\partial C^n} &= \left(\frac{\partial}{\partial C} + \frac{B}{A}\frac{\partial}{\partial D}\right)^n \Phi_1 \\
\frac{\partial^n \phi_1}{\partial \widetilde{B}^n} &= \left(\frac{1}{C}\frac{\partial}{\partial B}\right)^n \Phi_1,
\end{aligned}$$

and the right hand sides of these equalities vanish for n sufficiently large, since Φ_1 is a polynomial. Consequently, ϕ_1 is a polynomial in its arguments.

We decompose ϕ_1 as

$$\phi_1(A,C,\widetilde{B}) = AQ_1(A,C,\widetilde{B}) + CQ_2(A,C,\widetilde{B}) + Q_3(\widetilde{B}),$$

where Q_j are polynomials in their arguments. Notice that this decomposition is not unique. Now we set

$$\Phi_2(A,B,C,D) = BQ_1(A,C,\widetilde{B}) + DQ_2(A,C,\widetilde{B}) + \frac{B}{A}Q_3(\widetilde{B}) + \frac{1}{A}\widetilde{\Phi}_2(A,B,C,D) \tag{C.7}$$

so that the second equation in (C.6) leads to

$$\mathscr{D}^*\widetilde{\Phi}_2 = 0,$$

where $\widetilde{\Phi}_2$ is a polynomial in its arguments. The arguments used for ϕ_1 above, imply that

$$\widetilde{\Phi}_2(A,B,C,D) = \phi_2(A,C,\widetilde{B}),$$

with ϕ_2 polynomial in its arguments. Now multiplying (C.7) by A and taking $A = 0$ gives

$$BQ_3(BC) + \phi_2(0,C,BC)$$

for any $(B,C) \in \mathbb{R}^2$. This proves that

$$Q_3 = 0, \quad \phi_2(0,C,BC) = 0.$$

As a consequence, since B and BC are independent variables, the polynomial $\phi_2(A,C,\widetilde{B})$ is divisible by A, and there is a polynomial Q_4 such that

$$\Phi_2(A,B,C,D) = BQ_1(A,C,\widetilde{B}) + DQ_2(A,C,\widetilde{B}) + Q_4(A,C,\widetilde{B}).$$

Finally, notice that we can write

$$\begin{aligned} Q_4(A,C,\widetilde{B}) &= P_3(A,C) + \widetilde{B}Q_5(A,C,\widetilde{B}) \\ &= P_3(A,C) + BCQ_5(A,C,\widetilde{B}) - DAQ_5(A,C,\widetilde{B}), \end{aligned}$$

hence we obtain the final form

$$\begin{aligned} \Phi_1(A,B,C,D) &= AP_1(A,C,\widetilde{B}) + CP_2(A,C,\widetilde{B}) \\ \Phi_2(A,B,C,D) &= BP_1(A,C,\widetilde{B}) + DP_2(A,C,\widetilde{B}) + P_3(A,C), \end{aligned} \tag{C.8}$$

where

$$P_1 = Q_1 + CQ_5, \quad P_2 = Q_2 - AQ_5.$$

In the same way, from the last two equalities in (C.6) we obtain

$$\begin{aligned} \Phi_3(A,B,C,D) &= AP_4(A,C,\widetilde{B}) + CP_5(A,C,\widetilde{B}) \\ \Phi_4(A,B,C,D) &= BP_4(A,C,\widetilde{B}) + DP_5(A,C,\widetilde{B}) + P_6(A,C), \end{aligned} \tag{C.9}$$

which completes the proof of Lemma 1.19. □

C.5 Proof of Theorem 2.2 (Perturbed Normal Forms)

We have to determine two polynomials $\boldsymbol{\Phi}_\mu$ and $\mathbf{N}_\mu$ of degree p in $\mathbb{R}^n$ with coefficients depending upon μ, which satisfy the equality

$$\mathscr{F}(\boldsymbol{\Phi}_\mu, \mathbf{N}_\mu, \mu) = 0. \tag{C.10}$$

With the notations from Hypothesis 2.1, the map $\mathscr{F} : \mathscr{H}^2 \times \mathscr{V}_\mu \to \mathscr{H}$ defined by

$$(\mathscr{F}(\boldsymbol{\Phi},\mathbf{N},\mu))(v) \stackrel{\text{def}}{=} (\mathscr{A}_{\mathbf{L}}\boldsymbol{\Phi})(v) + \mathbf{N}(v) + \Pi_p(D\boldsymbol{\Phi}(v)\mathbf{N}(v) - \mathbf{R}(v+\boldsymbol{\Phi}(v),\mu)) \tag{C.11}$$

is of class $\mathscr{C}^k$, where we recall that $\mathscr{H}$ is the space of polynomials of degree p, and Π_p the linear map that associates to a map of class $\mathscr{C}^p$ the polynomial of degree p in its Taylor expansion. For notational simplicity, we suppress the indices μ and write $\boldsymbol{\Phi}$ and $\mathbf{N}$ instead of $\boldsymbol{\Phi}_\mu$ and $\mathbf{N}_\mu$, respectively.

For $\mu = 0$ we recover exactly the situation treated in Theorem 1.2. This means that we have a solution $(\boldsymbol{\Phi},\mathbf{N}) = (\boldsymbol{\Phi}^{(0)},\mathbf{N}^{(0)})$ of (C.10) for $\mu = 0$,

$$\mathscr{F}(\boldsymbol{\Phi}^{(0)},\mathbf{N}^{(0)},0) = 0,$$

which is unique when we restrict to

$$\boldsymbol{\Phi} \in (\ker \mathscr{A}_{\mathbf{L}})^{\perp}, \quad \mathbf{N} \in \ker(\mathscr{A}_{\mathbf{L}^*}). \tag{C.12}$$

In order to determine the polynomials $\boldsymbol{\Phi}$ and $\mathbf{N}$ for μ close to zero, we use the implicit function theorem to solve (C.10), together with (C.12), i.e., with

$$\mathscr{F} : (\ker \mathscr{A}_{\mathbf{L}})^{\perp} \times \ker(\mathscr{A}_{\mathbf{L}^*}) \times \mathscr{V}_\mu \to \mathscr{H}.$$

First, we compute the differential

$$\mathscr{D}_0 \stackrel{\text{def}}{=} D_{(\boldsymbol{\Phi},\mathbf{N})}\mathscr{F}(\boldsymbol{\Phi}^{(0)},\mathbf{N}^{(0)},0) : (\ker \mathscr{A}_{\mathbf{L}})^{\perp} \times \ker(\mathscr{A}_{\mathbf{L}^*}) \to \mathscr{H}$$

of $\mathscr{F}$ with respect to $(\boldsymbol{\Phi},\mathbf{N})$ at $(\boldsymbol{\Phi}^{(0)},\mathbf{N}^{(0)},0)$. A direct calculation gives

$$\begin{aligned}(\mathscr{D}_0(\boldsymbol{\Psi},\mathbf{M}))(v) = {} & (\mathscr{A}_{\mathbf{L}}\boldsymbol{\Psi})(v) + \mathbf{M}(v) + \Pi_p\Big(D\boldsymbol{\Psi}(v)\mathbf{N}^{(0)}(v) \\ & + D\boldsymbol{\Phi}^{(0)}(v)\mathbf{M}(v) - D\mathbf{R}(v+\boldsymbol{\Phi}^{(0)}(v),0)\boldsymbol{\Psi}(v)\Big),\end{aligned} \tag{C.13}$$

and we prove now that this linear map is invertible.

Denote by π_q the linear map on $\mathscr{H}$, which associates to a polynomial $\mathbf{P}$ the polynomial $\mathbf{P}_q$ obtained by suppressing the monomials in $\mathbf{P}$ of degree different of q. With this notation, according to Theorem 1.2 we have that

$$(\boldsymbol{\Phi}_0^{(0)},\mathbf{N}_0^{(0)}) = (\boldsymbol{\Phi}_1^{(0)},\mathbf{N}_1^{(0)}) = 0,$$

since the polynomials $(\boldsymbol{\Phi}^{(0)},\mathbf{N}^{(0)})$ are at least quadratic. Identifying the homogeneous polynomials of degrees $0,\ldots,p$ on the right hand side of (C.13) we obtain, successively,

$$\begin{aligned}&(\mathscr{A}_{\mathbf{L}}\boldsymbol{\Psi}_0)(v) + \mathbf{M}_0(v),\\ &(\mathscr{A}_{\mathbf{L}}\boldsymbol{\Psi}_1)(v) + \mathbf{M}_1(v) + D\boldsymbol{\Phi}_2^{(0)}(v)\mathbf{M}_0 - D\mathbf{R}_2(v)\boldsymbol{\Psi}_0,\end{aligned}$$

$$
\begin{aligned}
&(\mathscr{A}_{\mathbf{L}}\Psi_2)(v)+\mathbf{M}_2(v)+D\Phi_3^{(0)}(v)\mathbf{M}_0+D\Phi_2^{(0)}(v)\mathbf{M}_1(v)-D\mathbf{R}_2(v)\Psi_1(v)\\
&\qquad\qquad -D^2\mathbf{R}_2(\Psi_0,\Phi_2^{(0)}(v))-D\mathbf{R}_3(v)\Psi_0,\\
&(\mathscr{A}_{\mathbf{L}}\Psi_p)(v)+\mathbf{M}_p(v)+\sum_{1\le l\le p}D\Phi_{l+1}^{(0)}(v)\mathbf{M}_{p-l}(v)+\sum_{1\le l\le p-1}D\Psi_{p-l}(v)\mathbf{N}_{l+1}^{(0)}(v)\\
&\qquad\qquad -\sum_{2\le q\le p}\pi_p\left(D\mathbf{R}_q(v+\Phi^{(0)}(v))\Psi(v)\right),
\end{aligned}
$$

where $\mathbf{R}_s(u)=D_u^s\mathbf{R}(0,0)(u^{(s)})/s!$. Now notice that for any degree q between 0 and p, the formulas above are of the form

$$\mathscr{A}_{\mathbf{L}}\Psi_q+\mathbf{M}_q+\mathbf{G}_q,$$

with $\mathbf{G}_q$ depending only upon $\mathbf{M}_j$ and Ψ_j with $j=0,\dots,q-1$. We have seen in the proof of Theorem 1.2 that the equation

$$\mathscr{A}_{\mathbf{L}}\Psi+\mathbf{M}=\mathbf{Q}$$

has a unique solution

$$\mathbf{M}=\mathbf{P}_{\ker(\mathscr{A}_{\mathbf{L}^*})}\mathbf{Q}\in\ker(\mathscr{A}_{\mathbf{L}^*}),\quad \Psi\in(\ker\mathscr{A}_{\mathbf{L}})^{\perp},$$

for any homogeneous polynomial $\mathbf{Q}$ of degree q. This implies that the differential $\mathscr{D}_0$ is invertible, and by the implicit function theorem we conclude that (C.10) has a unique solution $(\Phi_\mu,\mathbf{N}_\mu)$ satisfying (C.12). Furthermore the map $\mu\mapsto(\Phi_\mu,\mathbf{N}_\mu)$ is of class $\mathscr{C}^k$, which implies that the coefficients of monomials of degree q are functions of μ of class $\mathscr{C}^{k-q}$. This completes the proof of Theorem 2.2. □

Remark C.1 *In the case where $\mathbf{R}(\cdot,\mu)$ is linear, equation (C.10) is affine in Φ, and we can look directly for a solution $(\Phi,\mathbf{N})\in(\mathscr{L}(\mathbb{R}^n))^2$, i.e., in the space of polynomials of degree 1.*

D Reversible Bifurcations

The references in this section are to theorems, hypotheses, formulas, and remarks in Chapter 4.

D.1 0^{3+} *Normal Form in Infinite Dimensions*

We show below how to compute the principal coefficients in the normal form (2.9) when starting from an infinite-dimensional system

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,\mu), \tag{D.1}$$

just like the system (4.1) in Chapter 3. We assume that the parameter μ is real, and that the system (D.1) possesses a reversibility symmetry $\mathbf{S}$ and satisfies the hypotheses of Theorems 3.3 and 3.15 in Chapter 2. We further assume that the spectrum of the linear operator $\mathbf{L}$ is such that $\sigma_0 = \{0\}$, where 0 is an algebraically triple and geometrically simple eigenvalue, with a symmetric eigenvector ζ_0 such that

$$\mathbf{S}\zeta_0 = \zeta_0.$$

Then the three-dimensional reduced system satisfies the hypotheses of Lemma 2.4, so that its normal form is given by (2.9).

For the computation of the coefficients in the expansion of P we proceed as in Section 3.4, and as for the reversible bifurcations in Section 4.1. We start from the equality (4.3) in Chapter 3, in which we take $v_0 = A\zeta_0 + B\zeta_1 + C\zeta_2$, and then write

$$u = A\zeta_0 + B\zeta_1 + C\zeta_2 + \widetilde{\Psi}(A,B,C,\mu), \tag{D.2}$$

where $\widetilde{\Psi}$ takes values in $\mathscr{Y}$. With the notations from Section 3.2.3, we consider the Taylor expansions of $\mathbf{R}$ in (1.15) and of $\widetilde{\Psi}$,

$$\widetilde{\Psi}(A,B,C,\mu) = \sum_{1\le r+s+q+n\le p} A^r B^s C^q \mu^n \Psi_{rsqn}, \quad \Psi_{rsq0} = 0 \text{ for } r+s+q=1.$$

Using the reversibility symmetry we find that

$$\mathbf{R}_{r,q}((\mathbf{S}u)^{(r)}) = -\mathbf{S}\mathbf{R}_{r,q}(u^{(r)}), \quad \mathbf{S}\Psi_{rsqn} = (-1)^s \Psi_{rsqn}.$$

Now we identify the different powers of (A,B,C,μ) in the identity (4.4) in Chapter 3, which is here

$$\begin{aligned}(\partial_A\widetilde{\Psi})B + (\partial_B\widetilde{\Psi})C + (\zeta_1 + \partial_B\widetilde{\Psi})AP(A,\widetilde{B},\mu) + (\zeta_2 + \partial_C\widetilde{\Psi})BP(A,\widetilde{B},\mu)\\ = \mathbf{L}\widetilde{\Psi} + \mathbf{R}(A\zeta_0 + B\zeta_1 + C\zeta_2 + \widetilde{\Psi},\mu).\end{aligned}$$

This leads to the following equalities, found successively at orders μ, $A\mu$, $B\mu$, $C\mu$:

$$\begin{aligned}
0 &= \mathbf{L}\Psi_{0001} + \mathbf{R}_{0,1}, &\text{(D.3)}\\
a\zeta_1 &= \mathbf{L}\Psi_{1001} + \mathbf{R}_{1,1}\zeta_0 + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{0001}), &\text{(D.4)}\\
a\zeta_2 + \Psi_{1001} &= \mathbf{L}\Psi_{0101} + \mathbf{R}_{1,1}\zeta_1 + 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{0001}), &\text{(D.5)}\\
\Psi_{0101} &= \mathbf{L}\Psi_{0011} + \mathbf{R}_{1,1}\zeta_2 + 2\mathbf{R}_{2,0}(\zeta_2,\Psi_{0001}); &\text{(D.6)}
\end{aligned}$$

at orders A^2, AB, AC, BC, C^2:

$$\begin{aligned}
b\zeta_1 &= \mathbf{L}\Psi_{2000} + \mathbf{R}_{2,0}(\zeta_0,\zeta_0), &\text{(D.7)}\\
b\zeta_2 + 2\Psi_{2000} &= \mathbf{L}\Psi_{1100} + 2\mathbf{R}_{2,0}(\zeta_0,\zeta_1), &\text{(D.8)}
\end{aligned}$$

$$\Psi_{1100} = \mathbf{L}\Psi_{0200} + \mathbf{R}_{2,0}(\zeta_1, \zeta_1), \tag{D.9}$$
$$\Psi_{1100} = \mathbf{L}\Psi_{1010} + 2\mathbf{R}_{2,0}(\zeta_0, \zeta_2), \tag{D.10}$$
$$2\Psi_{0200} + \Psi_{1010} = \mathbf{L}\Psi_{0110} + 2\mathbf{R}_{2,0}(\zeta_1, \zeta_2), \tag{D.11}$$
$$\Psi_{0110} = \mathbf{L}\Psi_{0020} + \mathbf{R}_{2,0}(\zeta_2, \zeta_2); \tag{D.12}$$

at orders A^3, A^2B, AB^2, B^3, A^2C, ABC:

$$\begin{aligned}
d\zeta_1 + b\Psi_{1100} &= \mathbf{L}\Psi_{3000} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{2000}) \\
&\quad + \mathbf{R}_{3,0}(\zeta_0, \zeta_0, \zeta_0), && \text{(D.13)} \\
d\zeta_2 + 3\Psi_{3000} + 2b\Psi_{0200} + b\Psi_{1010} &= \mathbf{L}\Psi_{2100} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{1100}) \\
&\quad + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{2000}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_0\zeta_1), && \text{(D.14)} \\
c\zeta_1 + 2\Psi_{2100} + b\Psi_{0110} &= \mathbf{L}\Psi_{1200} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{0200}) \\
&\quad + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{1100}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \zeta_1), && \text{(D.15)} \\
c\zeta_2 + 2\Psi_{1200} &= \mathbf{L}\Psi_{0300} + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{0200}) \\
&\quad + \mathbf{R}_{3,0}(\zeta_1, \zeta_1, \zeta_1), && \text{(D.16)} \\
-2c\zeta_1 + \Psi_{2100} + b\Psi_{0110} &= \mathbf{L}\Psi_{2010} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{1010}) \\
&\quad + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{2000}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_0, \zeta_2), && \text{(D.17)} \\
-2c\zeta_2 + 2\Psi_{2010} + 2\Psi_{1200} + 2b\Psi_{0020} &= \mathbf{L}\Psi_{1110} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{0110}) \\
&\quad + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{1010}) + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{1100}) \\
&\quad + 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \zeta_2); && \text{(D.18)}
\end{aligned}$$

and at orders AC^2, B^2C, BC^2, C^3:

$$\begin{aligned}
\Psi_{1110} &= \mathbf{L}\Psi_{1020} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{0020}) + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{1010}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_2, \zeta_2), && \text{(D.19)} \\
\Psi_{1110} + 3\Psi_{0300} &= \mathbf{L}\Psi_{0210} + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{0110}) + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{0200}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \zeta_2), && \text{(D.20)} \\
2\Psi_{0210} &= \mathbf{L}\Psi_{0120} + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{0020}) + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{0110}) \\
&\quad + 3\mathbf{R}_{3,0}(\zeta_1, \zeta_2, \zeta_2), && \text{(D.21)} \\
\Psi_{0120} &= \mathbf{L}\Psi_{0030} + 2\mathbf{R}_{2,0}(\zeta_2, \Psi_{0020}) + \mathbf{R}_{3,0}(\zeta_2, \zeta_2, \zeta_2). && \text{(D.22)}
\end{aligned}$$

Notice that due to the symmetry properties of $\mathbf{R}_{p,q}$, ζ_0, ζ_1, ζ_2, and since $\mathbf{S}\Psi_{rsqn} = (-1)^s \Psi_{rsqn}$, we have that

$$\mathbf{S}\mathbf{R}_{0,1} = -\mathbf{R}_{0,1}, \quad \mathbf{S}\mathbf{R}_{1,1}\zeta_j = (-1)^{j+1}\mathbf{R}_{1,1}\zeta_j, \quad j = 0, 1, 2,$$

and

$$\mathbf{SR}_{2,0}(\zeta_j, \boldsymbol{\Psi}_{rsqn}) = (-1)^{j+s+1}\mathbf{R}_{2,0}(\zeta_j, \boldsymbol{\Psi}_{rsqn}), \quad j = 0, 1, 2,$$
$$\mathbf{SR}_{3,0}(\zeta_j, \zeta_k, \zeta_l) = (-1)^{j+k+l+1}\mathbf{R}_{3,0}(\zeta_j, \zeta_k, \zeta_l), \quad j, k, l \in \{0, 1, 2\}.$$

The solvability conditions for these equations are now obtained by taking the duality product of each equation with the vector ζ_2^* orthogonal to the range of $\mathbf{L}$. Proceeding as for the other examples, we find that this vector is given by

$$\zeta_2^* = \mathbf{P}_0^* \zeta_{02}^* \in \mathscr{X}^*,$$

where $\mathbf{P}_0^*$ is the adjoint of the projection $\mathbf{P}_0$ onto the three-dimensional space $\mathscr{E}_0$, and ζ_{02}^* is the eigenvector associated with the eigenvalue 0 of the adjoint of $\mathbf{L}_0$, the restriction of $\mathbf{L}$ to $\mathscr{E}_0$, satisfying $\langle \zeta_2, \zeta_2^* \rangle = 1$. In addition, we have that

$$\langle \zeta_0, \zeta_2^* \rangle = 0, \quad \langle \zeta_1, \zeta^* \rangle = 0, \quad \langle \zeta_2, \zeta_2^* \rangle = 1,$$

and since $\mathbf{S}\zeta_2 = \zeta_2$), that

$$\mathbf{S}^* \zeta_2^* = \zeta_2^*.$$

We can now solve the system (D.3)–(D.22). Since any antisymmetric vector of $\mathscr{X}$ lies in the range of $\mathbf{L}$, it is straightforward to check that there is no solvability condition for the equations (D.3), (D.4), (D.6), (D.7), (D.9), (D.10), (D.12), and (D.13), (D.15), (D.17), (D.19), (D.20), (D.22). The solvability conditions for the remaining equations allow us to determine the coefficients a, c, and d.

First, the equation (D.3) gives $\boldsymbol{\Psi}_{0001}$, defined up to an arbitrary multiple of ζ_0. Then from (D.4) we obtain

$$\boldsymbol{\Psi}_{1001} = \widetilde{\boldsymbol{\Psi}}_{1001} + a\zeta_2,$$

where

$$\mathbf{L}\widetilde{\boldsymbol{\Psi}}_{1001} + \mathbf{R}_{1,1}\zeta_0 + 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{0001}) = 0.$$

For (D.5) we find the solvability condition

$$2a = \langle \mathbf{R}_{1,1}\zeta_1 + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{0001}) - \widetilde{\boldsymbol{\Psi}}_{1001}, \zeta_2^* \rangle,$$

which gives a, and from this equation we can determine $\boldsymbol{\Psi}_{0101}$ up to an arbitrary multiple of ζ_0. From (D.6) we can find $\boldsymbol{\Psi}_{0011}$ up to an arbitrary multiple of ζ_0, again.

Next, equation (D.7) gives

$$\boldsymbol{\Psi}_{2000} = \widetilde{\boldsymbol{\Psi}}_{2000} + b\zeta_2 + \psi_{2000}\zeta_0,$$

where

$$\mathbf{L}\widetilde{\boldsymbol{\Psi}}_{2000} + \mathbf{R}_{2,0}(\zeta_0, \zeta_0) = 0.$$

For (D.8) we find the solvability condition

$$3b = \langle 2\mathbf{R}_{2,0}(\zeta_0, \zeta_1) - 2\widetilde{\boldsymbol{\Psi}}_{2000}, \zeta_2^* \rangle,$$

which gives b, and we can determine

$$\boldsymbol{\Psi}_{1100} = \widetilde{\boldsymbol{\Psi}}_{1100} + 2\psi_{2000}\zeta_1,$$

where

$$\mathbf{L}\widetilde{\boldsymbol{\Psi}}_{1100} + 2\mathbf{R}_{2,0}(\zeta_0, \zeta_1) = 2\widetilde{\boldsymbol{\Psi}}_{2000}.$$

Similarly, from (D.9) and (D.10) we obtain,

$$\boldsymbol{\Psi}_{0200} = \widetilde{\boldsymbol{\Psi}}_{0200} + 2\psi_{2000}\zeta_2, \quad \boldsymbol{\Psi}_{1010} = \widetilde{\boldsymbol{\Psi}}_{1010} + 2\psi_{2000}\zeta_2,$$

where

$$\mathbf{L}\widetilde{\boldsymbol{\Psi}}_{0200} + \mathbf{R}_{2,0}(\zeta_1, \zeta_1) = \widetilde{\boldsymbol{\Psi}}_{1100}, \quad \mathbf{L}\widetilde{\boldsymbol{\Psi}}_{1010} + 2\mathbf{R}_{2,0}(\zeta_0, \zeta_2) = \widetilde{\boldsymbol{\Psi}}_{1100}.$$

Now for (D.11) we find the solvability condition

$$6\psi_{2000} = \langle 2\mathbf{R}_{2,0}(\zeta_1, \zeta_2) - 2\widetilde{\boldsymbol{\Psi}}_{0200} - \widetilde{\boldsymbol{\Psi}}_{1010}, \zeta_2^* \rangle,$$

and from (D.11) and (D.12) we can determine $\boldsymbol{\Psi}_{0110}$ and $\boldsymbol{\Psi}_{0020}$.
Finally, from (D.13) and (D.14) we find

$$\boldsymbol{\Psi}_{3000} = \widetilde{\boldsymbol{\Psi}}_{3000} + d\zeta_2 + \psi_{3000}\zeta_0, \quad \boldsymbol{\Psi}_{2100} = \widetilde{\boldsymbol{\Psi}}_{2100} + 3\psi_{3000}\zeta_1,$$

and the coefficient d,

$$\begin{aligned} 4d = & \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{1100}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{2000}) + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_0\zeta_1), \zeta_2^* \rangle \\ & - \langle 3\widetilde{\boldsymbol{\Psi}}_{3000} + 2b\boldsymbol{\Psi}_{0200} + b\boldsymbol{\Psi}_{1010}, \zeta_2^* \rangle. \end{aligned} \tag{D.23}$$

Now (D.15) and (D.16) give

$$\boldsymbol{\Psi}_{1200} = \widetilde{\boldsymbol{\Psi}}_{1200} + (c + 6\psi_{3000})\zeta_2, \quad \boldsymbol{\Psi}_{0300} = \widetilde{\boldsymbol{\Psi}}_{0300},$$

and

$$3c + 12\psi_{3000} = \langle 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{0200}) + \mathbf{R}_{3,0}(\zeta_1, \zeta_1, \zeta_1) - 2\widetilde{\boldsymbol{\Psi}}_{1200}, \zeta_2^* \rangle.$$

From (D.17) and (D.18) we obtain

$$\boldsymbol{\Psi}_{2010} = \widetilde{\boldsymbol{\Psi}}_{2010} + (3\psi_{3000} - 2c)\zeta_2 + \psi_{2010}\zeta_0, \quad \boldsymbol{\Psi}_{1110} = \widetilde{\boldsymbol{\Psi}}_{1110} + 2\psi_{2010}\zeta_1,$$

and

$$\begin{aligned} 12\psi_{3000} = & \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{0110}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{1010}) + 2\mathbf{R}_{2,0}(\zeta_2, \boldsymbol{\Psi}_{1100}), \zeta_2^* \rangle \\ & + \langle 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \zeta_2), \zeta_2^* \rangle - \langle 2\widetilde{\boldsymbol{\Psi}}_{2010} + 2\widetilde{\boldsymbol{\Psi}}_{1200} + 2b\boldsymbol{\Psi}_{0020}, \zeta_2^* \rangle. \end{aligned}$$

We conclude that

$$\begin{aligned} 3c = & \langle 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{0200}) - 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{0110}) - 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{1010}),\zeta_2^*\rangle \\ & -\langle 2\mathbf{R}_{2,0}(\zeta_2,\Psi_{1100}) - \mathbf{R}_{3,0}(\zeta_1,\zeta_1,\zeta_1) + 6\mathbf{R}_{3,0}(\zeta_0,\zeta_1,\zeta_2),\zeta_2^*\rangle \\ & +\langle 2\widetilde{\Psi}_{2010} + 2b\Psi_{0020},\zeta_2^*\rangle. \end{aligned} \tag{D.24}$$

Furthermore, from the equations (D.19) and (D.20) we can find

$$\Psi_{1020} = \widetilde{\Psi}_{1020} + 2\psi_{2010}\zeta_2, \quad \Psi_{0210} = \widetilde{\Psi}_{0210} + 2\psi_{2010}\zeta_2,$$

the solvability condition for (D.21) gives

$$4\psi_{2010} = \langle 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{0020}) + 2\mathbf{R}_{2,0}(\zeta_2,\Psi_{0110}) + 3\mathbf{R}_{3,0}(\zeta_1,\zeta_2,\zeta_2),\zeta_2^*\rangle,$$

and from (D.21) and (D.22) we can also determine Ψ_{0120} and Ψ_{0030}.

D.2 $(i\omega)^2$ Normal Form in Infinite Dimensions

We show below how to compute the principal coefficients in the normal form (3.25) when starting from an infinite-dimensional system of the form (3.12). We assume that the parameter μ is real and that the system (3.12) possesses a reversibility symmetry $\mathbf{S}$ and satisfies the hypotheses of Theorems 3.3 and 3.15 in Chapter 2. We further assume that the spectrum of the linear operator $\mathbf{L}$ is such that $\sigma_0 = \{\pm i\omega\}$, where $\pm i\omega$ are algebraically double and geometrically simple eigenvalues. Then the four-dimensional reduced system satisfies the hypotheses of Lemma 3.17, so that its normal form is given by (3.25).

We proceed as in Section 3.4, and in the previous cases. In equality (4.3) in Chapter 3, we take $v_0 = A\zeta_0 + B\zeta_1 + \overline{A}\overline{\zeta_0} + \overline{B}\overline{\zeta_1}$, and then write

$$u = A\zeta_0 + B\zeta_1 + \overline{A}\overline{\zeta_0} + \overline{B}\overline{\zeta_1} + \widetilde{\Psi}(A,B,\overline{A},\overline{B},\mu) \tag{D.25}$$

where $\widetilde{\Psi}$ takes values in $\mathscr{Z}$. With the notations from Section 3.2.3, we consider the Taylor expansion (1.15) of $\mathbf{R}$, and the expansion of $\widetilde{\Psi}$,

$$\widetilde{\Psi}(A,B,\overline{A},\overline{B},\mu) = \sum_{1\leq r+s+q+l+m\leq p} A^r B^s \overline{A}^q \overline{B}^l \mu^m \Psi_{rsqlm},$$

where

$$\Psi_{rsql0} = 0, \quad \text{for } r+s+q+l = 1.$$

Using the reversibility symmetry we find that

$$\widetilde{\Psi}(\overline{A},-\overline{B},A,B,\mu) = \mathbf{S}\widetilde{\Psi}(A,B,\overline{A},\overline{B},\mu),$$

and

$$\mathbf{S}\Psi_{rsqlm} = (-1)^{s+l}\Psi_{qlrsm}, \quad \Psi_{rsqlm} = \overline{\Psi}_{qlrsm}.$$

Identity (4.4) in Chapter 3, is in this case

$$
\begin{aligned}
&(i\omega A+B)\partial_A\Psi + i\omega B\partial_B\Psi + (-i\omega\overline{A}+\overline{B})\partial_{\overline{A}}\Psi - i\omega\overline{B}\partial_{\overline{B}}\Psi\\
&+\Big(iA(\zeta_0+\partial_A\Psi) - i\overline{A}(\overline{\zeta}_0+\partial_{\overline{A}}\Psi)\Big)P\\
&+(\zeta_1+\partial_B\Psi)(iBP+AQ)+(\overline{\zeta}_1+\partial_{\overline{B}}\Psi)(-i\overline{B}P+\overline{A}Q)\\
&\qquad\qquad = \mathbf{L}\Psi + \mathbf{R}(A\zeta_0+B\zeta_1+\overline{A}\overline{\zeta_0}+\overline{B}\overline{\zeta_1}+\Psi,\mu).
\end{aligned}
$$

Using the expansions of $\mathbf{R}$, $\widetilde{\Psi}$, P, and Q, we find at orders μ, $A\mu$, and $B\mu$, the equalities

$$
\begin{aligned}
0 &= \mathbf{L}\Psi_{00001} + \mathbf{R}_{0,1}, && \text{(D.26)}\\
a\zeta_1 + i\alpha\zeta_0 &= (\mathbf{L}-i\omega)\Psi_{10001} + \mathbf{R}_{1,1}\zeta_0 + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{00001}), && \text{(D.27)}\\
i\alpha\zeta_1 + \Psi_{10001} &= (\mathbf{L}-i\omega)\Psi_{01001} + \mathbf{R}_{1,1}\zeta_1 + 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{00001}), && \text{(D.28)}
\end{aligned}
$$

and at orders A^2, $A\overline{A}$, AB, $A\overline{B}$, B^2, $B\overline{B}$ we have:

$$
\begin{aligned}
0 &= (\mathbf{L}-2i\omega)\Psi_{20000} + \mathbf{R}_{2,0}(\zeta_0,\zeta_0), && \text{(D.29)}\\
0 &= \mathbf{L}\Psi_{10100} + 2\mathbf{R}_{2,0}(\zeta_0,\overline{\zeta_0}), && \text{(D.30)}\\
2\Psi_{20000} &= (\mathbf{L}-2i\omega)\Psi_{11000} + 2\mathbf{R}_{2,0}(\zeta_0,\zeta_1), && \text{(D.31)}\\
\Psi_{10100} &= \mathbf{L}\Psi_{10010} + 2\mathbf{R}_{2,0}(\zeta_0,\overline{\zeta_1}), && \text{(D.32)}\\
\Psi_{11000} &= (\mathbf{L}-2i\omega)\Psi_{02000} + \mathbf{R}_{2,0}(\zeta_1,\zeta_1), && \text{(D.33)}\\
\Psi_{10010}+\Psi_{01100} &= \mathbf{L}\Psi_{01010} + 2\mathbf{R}_{2,0}(\zeta_1,\overline{\zeta_1}). && \text{(D.34)}
\end{aligned}
$$

At orders A^3, A^2B, AB^2, and B^3 we obtain:

$$
\begin{aligned}
0 &= (\mathbf{L}-3i\omega)\Psi_{30000} + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{20000}) + \mathbf{R}_{3,0}(\zeta_0,\zeta_0,\zeta_0), && \text{(D.35)}\\
3\Psi_{30000} &= (\mathbf{L}-3i\omega)\Psi_{21000} + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{20000})\\
&\quad +3\mathbf{R}_{3,0}(\zeta_0,\zeta_0,\zeta_1), && \text{(D.36)}\\
2\Psi_{21000} &= (\mathbf{L}-3i\omega)\Psi_{12000} + 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{11000}) + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{02000})\\
&\quad +3\mathbf{R}_{3,0}(\zeta_0,\zeta_1,\zeta_1), && \text{(D.37)}\\
\Psi_{12000} &= (\mathbf{L}-3i\omega)\Psi_{03000} + 2\mathbf{R}_{2,0}(\zeta_1,\Psi_{02000}) + \mathbf{R}_{3,0}(\zeta_1,\zeta_1,\zeta_1), && \text{(D.38)}
\end{aligned}
$$

and finally at orders $A^2\overline{A}$, $A^2\overline{B}$, $A\overline{A}B$, $\overline{A}B^2$, $AB\overline{B}$, and $B^2\overline{B}$ we find:

$$
\begin{aligned}
b\zeta_1 + i\beta\zeta_0 &= (\mathbf{L}-i\omega)\Psi_{20100} + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{10100})\\
&\quad +2\mathbf{R}_{2,0}(\overline{\zeta_0},\Psi_{20000}) + 3\mathbf{R}_{3,0}(\zeta_0,\zeta_0,\overline{\zeta_0}), && \text{(D.39)}\\
\frac{ic}{2}\zeta_1 - \frac{\gamma}{2}\zeta_0 + \Psi_{20100} &= (\mathbf{L}-i\omega)\Psi_{20010} + 2\mathbf{R}_{2,0}(\zeta_0,\Psi_{10010})\\
&\quad +2\mathbf{R}_{2,0}(\overline{\zeta_1},\Psi_{20000}) + 3\mathbf{R}_{3,0}(\zeta_0,\zeta_0,\overline{\zeta_1}), && \text{(D.40)}
\end{aligned}
$$

$$\left(i\beta - \frac{ic}{2}\right)\zeta_1 + \frac{\gamma}{2}\zeta_0 + 2\Psi_{20100} = (\mathbf{L} - i\omega)\Psi_{11100} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{01100}) + 2\mathbf{R}_{2,0}(\overline{\zeta_0}, \Psi_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{10100}) + 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \overline{\zeta_0}), \tag{D.41}$$

$$\frac{\gamma}{2}\zeta_1 + \Psi_{11100} = (\mathbf{L} - i\omega)\Psi_{02100} + 2\mathbf{R}_{2,0}(\overline{\zeta_0}, \Psi_{02000}) + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{01100}) + 3\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \overline{\zeta_0}), \tag{D.42}$$

$$-\frac{\gamma}{2}\zeta_1 + 2\Psi_{20010} + \Psi_{11100} = (\mathbf{L} - i\omega)\Psi_{11010} + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{01010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \Psi_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{10010}) + 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \overline{\zeta_1}), \tag{D.43}$$

$$\Psi_{11010} + \Psi_{02100} = (\mathbf{L} - i\omega)\Psi_{02010} + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{01010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \Psi_{02000}) + 3\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \overline{\zeta_1}). \tag{D.44}$$

Notice that due to the symmetry properties of $\mathbf{R}_{p,q}$, ζ_0, and ζ_1 we have

$$\mathbf{S}\mathbf{R}_{0,1} = -\mathbf{R}_{0,1}, \quad \mathbf{S}\mathbf{R}_{1,1}\zeta_0 = -\overline{\mathbf{R}_{1,1}\zeta_0}, \quad \mathbf{S}\mathbf{R}_{1,1}\zeta_1 = \overline{\mathbf{R}_{1,1}\zeta_1},$$

which, together with the symmetry properties for Ψ_{qlrsm}, imply that applying the symmetry $\mathbf{S}$ to both sides of (D.27), (D.39), (D.42), and (D.43), we obtain the opposite of the complex conjugate of these equalities, whereas by applying it to (D.28), (D.40), (D.41), and (D.44), we find the complex conjugate of these equalities.

First, notice that the invertibility of the operators $\mathbf{L}$, $(\mathbf{L} - 2i\omega)$, and $(\mathbf{L} - 3i\omega)$, lets us solve the equations (D.26), (D.29)–(D.34), and (D.35)–(D.38), and determine $\Psi_{00001}, \Psi_{20000}, \Psi_{10100}, \Psi_{11000}, \Psi_{10010}, \Psi_{02000}, \Psi_{01010}, \Psi_{30000}, \Psi_{21000}, \Psi_{12000}$, and Ψ_{03000}.

Next, consider the vector ζ_1^* orthogonal to the range of $\mathbf{L} - i\omega$, constructed as in the other cases, such that

$$\langle \zeta_0, \zeta_1^* \rangle = 0, \quad \langle \overline{\zeta_0}, \zeta_1^* \rangle = 0, \quad \langle \zeta_1, \zeta_1^* \rangle = 1, \quad \langle \overline{\zeta_1}, \zeta_1^* \rangle = 0,$$

and

$$\mathbf{S}^*\zeta_1^* = -\overline{\zeta_1^*}.$$

Then from the equations (D.27) and (D.28) we find

$$a = \langle \mathbf{R}_{1,1}\zeta_0 + 2\mathbf{R}_{2,0}(\zeta_0, \Psi_{00001}), \zeta_1^* \rangle \tag{D.45}$$

and

$$\Psi_{10001} = \widetilde{\Psi}_{10001} + i\alpha\zeta_1,$$
$$2i\alpha = \langle \mathbf{R}_{1,1}\zeta_1 + 2\mathbf{R}_{2,0}(\zeta_1, \Psi_{00001}) - \widetilde{\Psi}_{10001}, \zeta_1^* \rangle. \tag{D.46}$$

Taking into account the fact that $\langle \mathbf{S}u, \mathbf{S}^*v \rangle = \langle u, v \rangle$ for any $u \in \mathscr{X}$ and $v \in \mathscr{X}^*$, it is not difficult to check that a and α are real numbers.

Next, equation (D.39) gives

$$b = \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{10100}) + 2\mathbf{R}_{2,0}(\overline{\zeta_0}, \boldsymbol{\Psi}_{20000}) + 3\mathbf{R}_{3,0}(\zeta_0, \zeta_0, \overline{\zeta_0}), \zeta_1^* \rangle, \tag{D.47}$$

and

$$\boldsymbol{\Psi}_{20100} = \widetilde{\boldsymbol{\Psi}}_{20100} + i\beta\zeta_1 + \psi_{20100}\zeta_0,$$

with a constant $\psi_{20100} \in \mathbb{R}$, which will be determined later. The equations (D.40) and (D.41) now give

$$\begin{aligned}
i\beta + \frac{ic}{2} &= \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{10010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \boldsymbol{\Psi}_{20000}), \zeta_1^* \rangle \\
&\quad + \langle 3\mathbf{R}_{3,0}(\zeta_0, \zeta_0, \overline{\zeta_1}) - \widetilde{\boldsymbol{\Psi}}_{20100}, \zeta_1^* \rangle, \\
3i\beta - \frac{ic}{2} &= \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{01100}) + 2\mathbf{R}_{2,0}(\overline{\zeta_0}, \boldsymbol{\Psi}_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{10100}), \zeta_1^* \rangle \\
&\quad + \langle 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \overline{\zeta_0}) - 2\widetilde{\boldsymbol{\Psi}}_{20100}, \zeta_1^* \rangle,
\end{aligned}$$

which determine the real coefficients β and c, and

$$\begin{aligned}
\boldsymbol{\Psi}_{20010} &= \widetilde{\boldsymbol{\Psi}}_{20010} + \left(\psi_{20100} - \frac{\gamma}{2}\right)\zeta_1, \\
\boldsymbol{\Psi}_{11100} &= \widetilde{\boldsymbol{\Psi}}_{11100} + \left(2\psi_{20100} + \frac{\gamma}{2}\right)\zeta_1 + \psi_{11100}\zeta_0,
\end{aligned}$$

with another constant ψ_{11100} determined later.

Finally, the equations (D.42)–(D.44) let us determine γ, ψ_{20100}, and ψ_{11100}. Indeed, (D.42) gives

$$\begin{aligned}
\gamma + 2\psi_{20100} &= \langle 2\mathbf{R}_{2,0}(\overline{\zeta_0}, \boldsymbol{\Psi}_{02000}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{01100}), \zeta_1^* \rangle \\
&\quad + \langle 3\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \overline{\zeta_0}) - \widetilde{\boldsymbol{\Psi}}_{11100}, \zeta_1^* \rangle,
\end{aligned}$$

and

$$\boldsymbol{\Psi}_{02100} = \widetilde{\boldsymbol{\Psi}}_{02100} + \psi_{11100}\zeta_1,$$

whereas (D.43) leads to

$$\begin{aligned}
4\psi_{20100} - \gamma &= \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{01010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \boldsymbol{\Psi}_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{10010}), \zeta_1^* \rangle \\
&\quad + \langle 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \overline{\zeta_1}) - 2\widetilde{\boldsymbol{\Psi}}_{20010} - \widetilde{\boldsymbol{\Psi}}_{11100}, \zeta_1^* \rangle,
\end{aligned}$$

and

$$\boldsymbol{\Psi}_{11010} = \widetilde{\boldsymbol{\Psi}}_{11010} + \psi_{11100}\zeta_1.$$

This determines γ and ψ_{20100}. In particular, we obtain

$$\begin{aligned}
3\gamma = {} & \langle 4\mathbf{R}_{2,0}(\overline{\zeta_0}, \boldsymbol{\Psi}_{02000}) + 4\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{01100}) + 6\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \overline{\zeta_0}), \zeta_1^* \rangle \\
& - \langle 2\mathbf{R}_{2,0}(\zeta_0, \boldsymbol{\Psi}_{01010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \boldsymbol{\Psi}_{11000}) + 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{10010}), \zeta_1^* \rangle \\
& - \langle 6\mathbf{R}_{3,0}(\zeta_0, \zeta_1, \overline{\zeta_1}) - 2\widetilde{\boldsymbol{\Psi}}_{20010} + \widetilde{\boldsymbol{\Psi}}_{11100}, \zeta_1^* \rangle.
\end{aligned}$$

Finally the equation (D.44) gives

$$\begin{aligned}2\psi_{11100} = &\langle 2\mathbf{R}_{2,0}(\zeta_1, \boldsymbol{\Psi}_{01010}) + 2\mathbf{R}_{2,0}(\overline{\zeta_1}, \boldsymbol{\Psi}_{02000}) + 3\mathbf{R}_{3,0}(\zeta_1, \zeta_1, \overline{\zeta_1}), \zeta_1^* \rangle \\ &- \langle \widetilde{\boldsymbol{\Psi}}_{11010} + \widetilde{\boldsymbol{\Psi}}_{02100}, \zeta_1^* \rangle,\end{aligned}$$

and completes the computation of the normal form.

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M. Haragus, G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*, Universitext,
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M. Haragus, G. Iooss, *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*, Universitext,
DOI 10.1007/978-0-85729-112-7,

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